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Optimal Mechanisms for Heterogeneous Multi-cell Aquifers

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1 Introduction

Over the past four decades a large economic literature has analyzed optimal aquifer management. An important assumption used in much of this work is that an aquifer behaves like a one-dimensional, single-cell "bathtub." In a single-cell model, water flows to the lowest point instantaneously and the water table is level throughout. Despite this assumption's mathematical convenience, aquifers are not underground caves filled with water, but rather saturated materials such as porous rock. As a result, transmissivity (horizontal flow) is lower than in a single-cell model and the water table elevation can vary across space.[1]

In a single-cell model, spatial considerations are unimportant. With limited transmissivity, however, location matters. Cones of depression develop around individual wells, and the impact of extraction on other users depends on the distance and geology between them. Water-extracting agents are not uniformly located on the land overlying the water reserves. Instead, they tend to be found in discrete clusters. Thus, a modeling assumption ignoring these effects can be expected to yield results of questionable validity, exaggerating the effects of an agent's extraction on distant users while understating effects on his immediate neighbors.

Here, we analyze the implications of relaxing this restriction. Instead of a single-cell, we use a multi-cell approximation similar to that commonly used in hydrological models such as MODFLOW (Harbaugh, 2005). Unsurprisingly, the more realistic dynamics come at a significant computational cost. Although we derive a mechanism that induces the socially optimal extraction path in Markov-perfect equilibrium for a general aquifer, it is likely to be too complicated to be implementable in practice. We are able to identify a class of aquifers, however, for which the social optimum can be induced through a remarkably simple linear

[1]See Brozović et al. (2006) for a thorough discussion of these issues.

pricing scheme. We conduct an illustrative simulation based on Indian aquifer data that suggests that a policy based on a false single-cell assumption can have significant adverse welfare impacts, even in the simple case.

Policy prescriptions incorrectly based on a single-cell model are clearly economically inefficient. One may suppose that this loss in efficiency may not be too problematic since the error would be on the side of being too cautious; after all, a bathtub assumption would tend to exaggerate the externalities generated by any user. This intuition is not necessarily true, however, if users are not homogeneous or are not uniformly spread through the aquifer. In our simulations we provide and discuss a simple example showing that an erroneous bathtub assumption may actually lead to *over*extraction for some or all users.

Literature abandoning the single-cell model in favor of more realistic dynamics has avoided strategic interaction among agents (Zeitouni and Dinar, 1997; Chakravorty and Umetsu, 2003; Brozović et al., 2010), and/or has assumed identical agents (Khalatbari, 1977; Eswaran and Lewis, 1984; Dasgupta, 2001; van der Ploeg, 2011).[2]

Our work incorporates another element overlooked in the previous literature: stock externalities. Groundwater extraction does not necessarily take place in environmental or political-economic isolation. The level of the groundwater stock may have costs felt beyond the users themselves. Environmental impacts may include effects on nearby wetlands, land subsidence, or saltwater intrusion in coastal areas. Political effects (our focus) may be felt if farmers' variable pumping costs are borne by the state. For example, it is common in many developing countries for agricultural electricity, the primary variable input to extraction, to be provided either for free or with a lump-sum tariff. India represents a salient example of the above policies and their complex (and at times controversial) economic and environmental effects (see World Bank, 2001; Dubash, 2007; Shah, 2008).[3] In general, the lower the water

[2]Assuming homogeneous agents greatly simplifies the analysis. All agents behave symmetrically, extracting the same amount of water in each period. Thus, if water tables levels are identical across space at time 0, there are no horizontal flows and hydrology is unimportant.

[3]Shah et al. (2007) estimate that annual Indian irrigation electricity costs are about $US 4 billion, relative to $US 29 billion to $36 billion of associated agricultural production.

table, the more energy is required to extract the water. A stock externality is thus generated to the extent that electricity costs are borne by the state rather than users.

We advance the literature by developing a spatially complex model of groundwater extraction addressing the above considerations.[4] Within this setting, we derive competitive equilibrium and socially-optimal extraction paths and investigate a class of policy instruments with relatively low informational requirements regarding user behavior. Specifically, since continuous monitoring of agent extraction rates is likely to be infeasible, we only allow governmental transfers to be made on the stock of the resource. In this sense, the regulatory problem is similar to that of dynamic non-point source pollution problems (e.g., Xepapadeas, 1992; Karp, 2005). In contrast to these earlier steady state analyses, however, we derive mechanisms that exactly induce the socially optimal extraction *path* in Markov-perfect equilibrium.[5]

As exceedingly complex instruments are unlikely to be appealing in practice, we initially restrict attention to simple linear additive mechanisms (a linear transfer based on a user's water table depth). For an interesting class of aquifers (those for which exhaustion is not a concern in the finite time frame being considered and for which the users bear negligible extraction costs) imposing such severe restrictions on policy instruments does not reduce social welfare. This positive result holds regardless of the number and spatial configuration of extracting agents as well as the spatial and physical characteristics of the aquifer (number of cells, connectivity).

We then show how the two characteristics of fully externalized state effects and inexhaustibility are crucial to ensuring the Markov-perfection of the linear mechanism. If either condition does not hold, the optimal transfer becomes more complicated, being a second-degree polynomial function of not only the user's water table depth, but that of every other

[4]Our results are still subject to such common simplifying assumptions as perfectly rational agents and commonly known deterministic hydrology.

[5]Game theoretic groundwater models using the bathtub model and identical users (e.g., Rubio and Casino, 2003) also have derived Markov perfect equilibrium paths that lead to the socially optimal steady state, without identifying paths that are optimal at *each* moment in time along the way.

user as well.

To illustrate the importance of allowing for spatial considerations we conduct comparative simulations based on aquifer characteristics in the Indian state of Andhra Pradesh. In a simple two-player game we derive the socially optimal extraction path and exhibit the linear transfer schemes that induce it in equilibrium. We further investigate the policy implications of improperly using a bathtub model and of failing to allow for agent heterogeneity. Our findings suggest that substantial welfare loss may result from implementing policy that is predicated on incorrect physical and technological assumptions.

The paper is organized as follows. In the next two sections we introduce our model and analyze the socially optimal solution. In section four, we first show how the linear mechanism induces the social optimum with zero equilibrium transfers for the special case of an inexhaustible aquifer with fully externalized extraction costs. We then derive the optimal mechanism for the more general (and complicated) case in which one or both of these conditions is violated. In section five we conduct a numerical simulation applying our theoretical results for the linear mechanism to a simple groundwater game set in rural India. We conclude with a discussion of results and useful extensions. All proofs are presented in the Appendix.

2 Hydrological Model

We model the aquifer as a common property resource not characterized by open access (the number and location of agents is fixed over time). The aquifer consists of N discrete interconnected cells, indexed by $n = 1, 2, ..., N$, each with a single user.[6]

Let $x_n(t) \in \Re_+$ denote elevation at time t for cell n. The extraction rate at time t for agent n is $q_n(t)$ and is restricted to lie in the interval $[0, \bar{q}]$, with $\bar{q}$ representing the (finite)

[6]For expository reasons, in this section we limit attention to one agent per cell, so the two terms are interchangeable. In such cases, it is theoretically possible to infer the pumping schedules of agents from the evolution of the water tables. Under the more general setting described in Footnote 10, however, such inference is not possible.

maximum technically feasible instantaneous extraction. The uniform rate of recharge is r, and a_n is the surface area of user n's land multiplied by the storativity coefficient of user n's micro-watershed (which depends on unspecified geological factors).[7] Parameter $c_{mn} \in [0, \infty]$ denotes the transmissivity between aquifer cells, a measure of the "connectivity" between the agents. It summarizes characteristics such as geological properties, saturated thickness of the intervening subsurface material, and distance between cells.[8] The term $c_{mn}[x_m - x_n]$ is the water flux between agent m and n's micro-watersheds.[9] Departing from the one-dimensional bathtub model, the water tables follow the set of differential equations (a dot indicates a derivative with respect to time):

$$a_n \cdot \dot{x}_n = r - q_n + \sum_{m=1}^{N} c_{mn}[x_m - x_n]. \tag{1}$$

Variations of these dynamics appear in Eswaran and Lewis (1984); Khalatbari (1977); Zeitouni and Dinar (1997).[10]

Solving the system of differential equations (1) yields water table levels at time t, as functions of initial conditions, $x_n(0)$, and the extraction history $q_n(s)$ for $0 \leq s \leq t$. Since the general solution is notationally unwieldy, we focus our discussion on the relatively simple example of two adjacent cells, indexed by n and m. For this case, letting $c_n \equiv c_{mn}/a_n = c_{nm}/a_n$, agent n's water table level at time t is:

$$\begin{aligned} x_n(t) \quad = \quad & \frac{1}{a_1 + a_2}\Big\{ x_n(0)\left[a_n + a_m \mathrm{e}^{-[c_1+c_2]t}\right] + x_m(0) a_m \left[1 - \mathrm{e}^{-[c_1+c_2]t}\right] + \\ & \int_0^t \left[[r - q_n(s)] \left[1 + \frac{a_m}{a_n} \mathrm{e}^{[c_1+c_2][s-t]}\right] + [r - q_m(s)] \left[1 - \mathrm{e}^{[c_1+c_2][s-t]}\right] \right] \mathrm{d}s \Big\}. \end{aligned} \tag{2}$$

[7]Our analysis extends to cases where recharge rates are different and may depend linearly on water tables.

[8]For non-adjacent cells $c_{mn} = 0$, while $c_{mn} = \infty$ for users in the same cell (as described in Footnote 10). We define $c_{nn} \equiv 0$.

[9]Using water balance and connectivity between individual cells with uniform recharge rates to model flows is a version of the finite difference discretization used in hydrological models such as MODFLOW that are commonly used to simulate aquifers with complex geometry and boundary conditions (Harbaugh, 2005).

[10]More generally, with multiple users per cell, Eq.(1) would read $a_n \cdot \dot{x}_n = r - \sum_{k \in A_n} q_k + \sum_{m=1}^{N} c_{mn}[x_m - x_n]$, where A_n denotes the set of agents occupying cell n.

Lemma 1 states that these dynamics nest the extreme cases of unconnected cells and a single-cell bathtub if the extraction rate is bounded and does not change too quickly.

Lemma 1 *Suppose $N = 2$ and let $q_n(t; c)$ and $x_n(t; c)$ denote the extraction rate and water table at time t as functions of connectivity, c. If extraction paths $q_n(t)$ are bounded and differentiable with bounded derivatives, water table dynamics defined by Eq. (1): (i) approach the isolated cells case, $\dot{x}_n = \frac{r - q_n(t;0)}{a_n}$ as $c \to 0$; and (ii) approach the single-cell case, $\dot{x}_n = \frac{2r - q_1(t;\infty) - q_2(t;\infty)}{a_1 + a_2}$ as $c \to \infty$.*

3 Social Optimum

We assume that each agent's gross (of energy costs) restricted profit function, $h_n(q_n(t))$, is concave in extraction.[11] The social planner maximizes the net benefit of water extraction: the discounted (at rate $\delta > 0$) sum of agent profit less energy cost, $D(\mathbf{q}(t), \mathbf{x}(t)) = \sum_{n=1}^{N} d(q_n(t), x_n(t))$. Here, $\mathbf{q}(t) \equiv (q_1(t), q_2(t), ..., q_N(t))'$, $\mathbf{x}(t) \equiv (x_1(t), x_2(t), ..., x_N(t))'$, and $d(q_n(t), x_n(t))$ represents the standard energy cost function

$$d(q_n(t), x_n(t)) = q_n[\omega_1 + \omega_2[\bar{x} - x_n]]. \tag{3}$$

For notational convenience, cost parameters $\omega_1 \geq 0$ and $\omega_2 \geq 0$, and ground level, $\bar{x}$, are assumed to be identical across users.

Initial conditions are $\mathbf{x_0} \equiv (x_{01}, x_{02}, ..., x_{0N})'$. The terminal time is T, and the "scrap value" of the aquifer is $-D^T(\mathbf{x}(T))$. The social planner's optimal control problem is

$$\begin{aligned} \max_{\mathbf{q}(t) \in [\mathbf{0}, \bar{\mathbf{q}}]} \quad & \int_0^T \mathrm{e}^{-\delta t} \left[\sum_{n=1}^{N} h_n(q_n(t)) - D(\mathbf{q}(t), \mathbf{x}(t)) \right] \mathrm{d}t - \mathrm{e}^{-\delta T} D^T(\mathbf{x}(T)) \\ \text{subject to:} \quad & \text{Eq. (1);} \\ & \mathbf{x} \geq \mathbf{0}; \;\; \mathbf{x}(0) = \mathbf{x_0}. \end{aligned} \tag{4}$$

[11] To maintain tractability in Section 4.2, we adopt a common convention in the groundwater literature (see Gisser and Sánchez, 1980; Rubio and Casino, 2003, among many others), restricting attention to quadratic functions.

We restrict attention to cases in which the social optimum does not involve exhaustion of the resource for any agent before time T. Letting $\boldsymbol{\lambda}(t) \equiv (\lambda_1(t), \lambda_2(t), ..., \lambda_N(t))'$ denote costate variables, the current-value Hamiltonian is

$$\begin{aligned} H(\mathbf{q}(t), \mathbf{x}(t), \boldsymbol{\lambda}(t), t) = \quad & \sum_{n=1}^{N} h_n(q_n(t)) - D(\mathbf{q}(t), \mathbf{x}(t)) \\ & + \sum_{n=1}^{N} \frac{\lambda_n(t)}{a_n} \left[r - q_n(t) + \sum_{m=1}^{N} c_{mn}[x_m(t) - x_n(t)] \right]. \end{aligned} \tag{5}$$

The necessary conditions for an interior solution are

$$\frac{\mathrm{d}h_n(q_n(t))}{\mathrm{d}q_n} - \frac{\partial D(\mathbf{q}(t), \mathbf{x}(t))}{\partial q_n} - \frac{\lambda_n(t)}{a_n} \quad = \quad 0. \tag{6}$$

Optimal conditions for the co-state variables yield the following differential equations

$$\dot{\lambda}_n(t) = \left[\delta + \sum_{m=1}^{N} \frac{c_{mn}}{a_n} \right] \lambda_n(t) - \sum_{m=1}^{N} \frac{c_{mn}}{a_m} \lambda_m(t) + \frac{\partial D(\mathbf{q}(t), \mathbf{x}(t))}{\partial x_n}. \tag{7}$$

Transversality conditions $\lambda_n(T) = -\partial D^T(x_n(T))/\partial x_n$ imply

$$\frac{\mathrm{d}h_n(q_n(T))}{\mathrm{d}q_n} = \frac{\partial D(\mathbf{q}(T), \mathbf{x}(T))}{\partial q_n} + \frac{\partial D^T(\mathbf{x}(T))}{\partial x_n}, \tag{8}$$

providing terminal conditions for extraction rates and water table levels.

Conditions (6), (7), and (8) are necessary and sufficient for a strictly interior optimum in which the aquifer is not exhausted (see Sethi and Thompson, 2000). Differentiating Eq. (6) with respect to t yields,

$$\frac{\dot{\lambda}_n(t)}{a_n} = \frac{\mathrm{d}^2 h_n(q_n(t))}{\mathrm{d}q_n^2} \dot{q}_n(t) - \sum_{m=1}^{N} \frac{\partial^2 D(\mathbf{q}(t), \mathbf{x}(t))}{\partial q_n \partial q_m} \dot{q}_m(t) - \sum_{m=1}^{N} \frac{\partial^2 D(\mathbf{q}(t), \mathbf{x}(t))}{\partial q_n \partial x_m} \dot{x}_m(t). \tag{9}$$

Substituting Eqs. (6) and (9) into (7), and rewriting the stock dynamics given by Eq. (1), we obtain N^2 differential equations involving $\mathbf{q}(t)$ and $\mathbf{x}(t)$. This system, together with initial conditions on the water stocks and terminal conditions on the extraction rates, specifies the socially optimal extraction and water stock paths $\langle \mathbf{q}^{SO}(t), \mathbf{x}^{SO}(t) \rangle$.

One way to achieve the social optimum would be for the government to mandate water extraction by each user. Of course, governments rarely have such sweeping power, or even the ability to monitor individual extraction decisions. In the next section, we derive decentralized pricing schemes based on aquifer depth (rather than individual extraction rates) that induce users to undertake the socially optimal extraction path in Markov perfect equilibrium.

We model the solution under two different assumptions regarding the size of the aquifer and extraction costs. First, we assume that the aquifer is sufficiently large that it cannot physically be exhausted before time T, and that all extraction costs are external to the users. Under this framework, the government can use a simple pricing scheme for each user that is a linear function of the water level in only that user's cell.

We next relax the assumption of inexhaustibility. The solution to this problem is more complex, with the pricing scheme becoming a second-degree polynomial function of the water levels in all cells. Finally, we allow for some extraction costs to be internalized by the agents in an exhaustible aquifer. The resulting polynomial pricing scheme is qualitatively similar to that of the exhaustible case with no internalized extraction costs.

4 Optimal Pricing Mechanisms

In this section, we suppose the regulator does not have resources to monitor agents' extraction decisions, but can costlessly monitor the state variables $\mathbf{x}(t)$. This scenario is analogous to a dynamic nonpoint source stock pollution problem in which the regulator can monitor ambient pollution levels but not individual emissions (e.g., Xepapadeas, 1992). The regulator's problem is how to design a pricing scheme based only upon water table depth that can induce users to undertake socially optimal extraction in Markov perfect equilibrium.

A mechanism $\boldsymbol{\phi}(\mathbf{x}(t), t) \equiv (\phi_1(x_1(t), t), \phi_2(x_2(t), t), ..., \phi_N(x_N(t), t))'$ is a vector of agent-specific transfers that depend upon the water table level in a user's cell. Such a mechanism induces a differential game between the users. A user's instantaneous profit is $h_n(q_n(t)) - \mu d(q_n(t), x_n(t)) + \phi_n(x_n(t), t)$, where μ indicates the proportion of energy costs borne by the user.[12] Given a strategy profile $q_m^*(\mathbf{x}(t),t)$ chosen by users $m \neq n$, user n chooses his strategy to solve:

$$\begin{aligned}
&\max_{q_n(t)\in[0,\bar{q}]} && \int_0^T \{h_n(q_n(t)) - \mu d(q_n(t), x_n(t)) + \phi_n(x_n(t),t)\}\,\mathrm{e}^{-\delta t}\mathrm{d}t + \phi_n(x_n(T),T)\mathrm{e}^{-\delta T} \\
&\text{subject to:} && a_n\dot{x}_n(t) = r - q_n(t) + \sum_{m=1}^{N} c_{mn}[x_m(t) - x_n(t)]; \\
& && a_m\dot{x}_m(t) = r - q_m^*(\mathbf{x}(t),t) + \sum_{\ell=1}^{N} c_{\ell m}[x_\ell(t) - x_m(t)] \ \text{ for all } m \neq n; \\
& && \mathbf{x} \geq \mathbf{0}; \ \ \mathbf{x}(0) = \mathbf{x_0}.
\end{aligned} \tag{10}$$

Let $X(\mathbf{x_0},t)$ be the set of all water table depths that can possibly be attained by time t starting from initial condition $\mathbf{x_0}$, given the restriction that extraction rates lie in the interval $[0,\bar{q}]$. An open-loop strategy is one in which users pre-commit to an entire extraction path at the beginning of the game, and so is not a function of current state variables. With a slight abuse of notation, let $\mathbf{q}^*(t)$ denote a strategy that depends only on time, not the vector of state variable, $\mathbf{x}$. Formally, a strategy $q_n^*(\mathbf{x}(t),t)$ is open-loop, if $q_n^*(\mathbf{x}(t),t) = q_n^*(t)$ for all $\mathbf{x}(t) \in X(\mathbf{x_0},t) \subseteq \Re_+^N$. An open-loop Nash equilibrium (defined below) is relatively simple to compute for this game.

Definition 1 *A vector $\mathbf{q}^*(t)$ of open-loop strategies where $q_n^*(t) : [0,T] \mapsto [0,\bar{q}]$, is an open-loop Nash equilibrium if, for each $n = 1, 2, ..., N$ an optimal control path $q_n(t)$ of the maximization problem given by (10) exists and is given by $q_n(t) = q_n^*(t)$.*

In general, open-loop Nash equilibria are restrictive since they do not allow users to adapt strategies to changes in the state vector. This equilibrium concept is typically justifiable only if the state vector is unobservable over time, rendering moot the ability to adapt.

Markovian equilibrium is a more flexible concept. A user choosing a Markov strategy conditions his current extraction only on the value of the current state variable (not otherwise

[12] The analysis easily extends to the case in which each user bears a different proportion of extraction costs.

on the game's previous history).[13]

Definition 2 *Let* $\mathbf{x}(t) \in X(\mathbf{x_0}, t)$ *for all* $t \in [0, T]$*. A vector* $\mathbf{q}^*(\mathbf{x}(t), t)$ *of Markovian strategies, where* $q_n^*(\mathbf{x}(t), t) : X(\mathbf{x_0}, t) \times [0, T] \mapsto [0, \bar{q}]$*, is a Markovian-Nash equilibrium if, for each* $n = 1, 2, ..., N$ *an optimal control path* $q_n(t)$ *of the maximization problem given by (10) exists and is given by* $q_n(t) = q_n^*(\mathbf{x}(t), t)$*.*

On its own, Markovian-Nash equilibrium does not provide much analytical advantage over open-loop equilibrium since all open loop equilibria are also (but not necessarily the only) Markovian-Nash equilibria (Dockner et al., 2000). Although Markovian-Nash equilibrium strategies are time consistent (in the sense that no user would have an incentive to unilaterally deviate from his strategy on the equilibrium path), they may not be credible. The lack of credibility arises due to the fact that if one user deviates, he may rationally expect other users to deviate as well in reaction to his deviation. The Markovian-Nash equilibrium concept does not rule out cases in which a user may profitably deviate from the equilibrium path due to his rational expectation of how his deviation will cause other players to update their strategies.

A Markov-perfect equilibrium is a Markovian-Nash equilibrium that avoids the above concerns. It is not vulnerable to the credibility problem since equilibrium strategies are defined over every possible sub-game, even those off the equilibrium path.

Definition 3 *A Markov-perfect equilibrium is a subgame-perfect Markovian-Nash equilibrium.*

Identifying a Markov-perfect equilibrium typically requires the solution of a complex system of Hamilton-Jacobi-Bellman equations. In the first case below, however, we show if extraction costs are purely external to the user and the state non-negativity constraint does not bind in any feasible subgame then the game has a structure that simplifies calculation

[13]See Chapter 4 of Dockner et al. (2000) for a detailed discussion of the role of informational assumptions in the determination of equilibrium strategies.

of Markov-perfect equilibria. Specifically, it is a linear state game (as defined by Dockner et al., 2000) since (a) its objective functionals and state dynamics are linear in the state, and (b) there are no cross terms of the sort $q_n x_n$ involving control and state variables. Dockner et al. (2000) (pp. 187-89) show that all open-loop Nash equilibria of linear state games are Markov-perfect.

To take advantage of this equivalence result in Section 4.1, we must make additional assumptions on our model primitives regarding the possibility of aquifer exhaustion. The reason for this is straightforward considering the nature of open loop equilibria. For an open-loop equilibrium to be Markov-perfect, strategies must be optimal for each player in every possible subgame, even off the equilibrium path. If a strategy involves extraction larger than natural recharge, it cannot be feasible (and thus cannot be optimal) in subgames for which the aquifer is exhausted.

In Section 4.1, we consider cases in which users do not incur extraction costs and it is not possible to exhaust the aquifer before time T. To ensure the latter, we assume that the range of initial water-table levels $\mathbf{x_0}$ is such that no cell in the aquifer is exhausted by time T even if all players extracted at the *maximum* possible rate $\bar{q}$ throughout the *entire* time horizon T. In our previous notation, this implies that we only consider a set of initial water tables X_0 such that

$$\mathbf{x_0} \in X_0 \iff X(\mathbf{x_0}, t) \subseteq \Re^N_{++} \text{ for all } t \in [0, T]. \tag{11}$$

lmplicit variations on this assumption are prevalent in the groundwater economics literature. A water-table non-negativity constraint is typically not explicitly imposed (Gisser and Sánchez, 1980; Fisher and Rubio, 1997; Zeitouni and Dinar, 1997; Roseta-Palma and Xepapadeas, 2004; Aggarwal and Narayan, 2004; Brozović et al., 2006, among others), or is modeled asymptotically allowing for finite (or even steady-state) violation at some parameter values (e.g., Rubio and Casino, 2003). Other studies either impose structure on model primitives that precludes socially optimal corner solutions (Negri, 1989), or deal with static water-table levels (Chakravorty and Umetsu, 2003).

In Section 4.2, we allow users to incur a positive fraction of their extraction costs and allow the aquifer to be exhausted. Unlike Section 4.1, we do not assume that initial water table levels are sufficiently high so as to prevent exhaustion. We therefore explicitly model a non-negativity constraint on $\mathbf{x}$. Either one of these conditions complicates the analysis by ruling out the linear-state game equilibrium equivalence result of Dockner et al. (2000); instead we employ standard dynamic programming techniques.

4.1 No resource exhaustion by T and no internal extraction cost

We first consider a case in which the user does not bear any extraction costs and the resource cannot become exhausted by time period T. Formally, we assume $\mu = 0$ and that the initial water table $\mathbf{x}_0$ belongs to X_0 as defined in Eq. (11). The externalization of extraction costs is similar to the stylized facts in countries such as India that do not impose tariffs on variable electricity usage.[14] In Theorem 1 below, we show that in spite of the complexity of the problem, the regulator can induce the socially optimal path in Markov perfect equilibrium with a surprisingly simple linear transfer mechanism. We define the linear mechanism as $\boldsymbol{\phi}^L(\mathbf{x}(t), t)$, such that

$$\phi_n^L(\mathbf{x}(t), t) = \begin{cases} \beta_n(t)[x_n(t) - x_n^{SO}(t)] & \text{for } t < T \\ \beta_n^T[x_n(t) - x_n^{SO}(t)] & \text{for } t = T. \end{cases} \tag{12}$$

Mechanism (12) is a simple instance of well-studied nonpoint source pollution control policies (e.g., Segerson, 1988; Xepapadeas, 1992; Athanassoglou, 2010). As exhaustion is not possible, the non-negativity constraint $\mathbf{x} \geq \mathbf{0}$ is trivially satisfied and therefore need not be imposed. The following theorem states that the linear mechanism $\boldsymbol{\phi}^L(\mathbf{x}(t), t)$ can induce any extraction path in Markov-perfect equilibrium.

[14] We abstract from the cost of drilling boreholes of different depths. For analysis of the problem of choosing strategic drilling strategies, see Aggarwal and Narayan (2004).

Theorem 1 *Let* $\left\{\hat{q}_n(t) : t \in [0,T]\right\}$, *be an arbitrary continuously differentiable feasible extraction path satisfying* $\frac{\mathrm{d}h_n(\hat{q}_n(t))}{\mathrm{d}q_n} < \infty$, *and* $q_n^{\boldsymbol{\phi}^L}(t)$ *be the unique Markov-perfect equilibrium in open-loop strategies that is induced by linear-state mechanism* $\boldsymbol{\phi}^L$. *If* $q_n^{\boldsymbol{\phi}^L}(t)$ *is everywhere interior, then there exists a unique mechanism,* $\boldsymbol{\phi}^L$, *such that* $q_n^{\boldsymbol{\phi}^L}(t) = \hat{q}_n(t)$ *for all* $t \in [0,T]$ *and* $n = 1, 2, ..., N$.

Although the general proof is stated in the Appendix, it is easiest to interpret the theorem for the simpler two-user case. We first note that a linear mechanism $\boldsymbol{\phi}^L(\mathbf{x}(t), t)$ induces a differential game (10) that has a unique Markov-perfect equilibrium in open loop strategies $\mathbf{q}^{\phi^L}(t)$.

To obtain this result, let $\boldsymbol{\lambda}_n(t) = (\lambda_{1n}, \lambda_{2n})'$ denote the costate variables for user n corresponding to the water tables of the two cells. For an open-loop Nash equilibrium, the current-value Hamiltonian of user n is:

$$\begin{aligned} H_n(q_n(t), \mathbf{x}(t), \boldsymbol{\lambda}_n(t), t) &= h_n(q_n) + \beta_n(t)[x_n(t) - x_n^{SO}(t)] + \\ &\frac{\lambda_{nn}(t)\left[r - q_n(t) + c_{mn}[x_m(t) - x_n(t)]\right]}{a_n} + \frac{\lambda_{mn}(t)\left[r - q_m(t) + c_{mn}[x_n(t) - x_m(t)]\right]}{a_m}, \end{aligned} \tag{13}$$

for $m, n = 1, 2$ and $m \neq n$. The necessary conditions for an interior solution are:[15]

$$\frac{\mathrm{d}h_n(q_n^{\phi^L}(t))}{\mathrm{d}q_n} = \frac{\lambda_{nn}(t)}{a_n} \tag{14}$$

$$\dot{\lambda}_{nn}(t) = [\delta + c_n]\lambda_{nn}(t) - c_m\lambda_{mn}(t) - \beta_n(t) \tag{15}$$

$$\dot{\lambda}_{mn}(t) = [\delta + c_m]\lambda_{mn}(t) - c_n\lambda_{nn}(t), \tag{16}$$

with transversality conditions

$$\lambda_{nn}(T) = \beta_n^T, \quad \lambda_{mn}(T) = 0. \tag{17}$$

Since $H_n(\cdot)$ is jointly concave in $q_n(t)$ and $\mathbf{x}(t)$, these conditions are also sufficient.

[15] Recall our earlier notation for the two-user case $\frac{c_{mn}}{a_n} \equiv c_n$.

Eqs. (15) and (16) are a linear system of ordinary differential equations. Imposing the transversality condition yields the unique solution for user n:

$$\lambda_{nn}(t) = \int_t^T \beta_n(s)\mathrm{e}^{\delta[t-s]}\frac{a_n + a_m \mathrm{e}^{[c_1+c_2][t-s]}}{a_1+a_2}\mathrm{d}s + \beta_n^T \mathrm{e}^{\delta[t-T]}\frac{a_n + a_m \mathrm{e}^{[c_1+c_2][t-T]}}{a_1+a_2}, \tag{18}$$

$$\lambda_{mn}(t) = \int_t^T \frac{\beta_n(s)\mathrm{e}^{\delta[t-s]}\left[1-\mathrm{e}^{[c_1+c_2][t-s]}\right]a_m}{a_1+a_2}\mathrm{d}s + \frac{\beta_n^T \mathrm{e}^{\delta[t-T]}\left[1-\mathrm{e}^{[c_1+c_2][t-T]}\right]a_m}{a_1+a_2}. \tag{19}$$

To understand the intuition behind this result, note that the state dynamics described in Eq. (2) can be used to derive the shadow value of a unit of water table height for user n at time t given a mechanism $\boldsymbol{\phi}^L(\mathbf{x}(t), t)$. Specifically, Eq. (2) implies for $s \in (t, T)$ and for $n, m = 1, 2$ with $n \neq m$,

$$x_n(s) = \frac{1}{a_1+a_2}\Bigg\{x_n(t)\left[a_n + a_m\mathrm{e}^{[c_1+c_2][t-s]}\right] + x_m(t)a_m\left[1-\mathrm{e}^{[c_1+c_2][t-s]}\right] + \tag{20}$$
$$\int_t^s \left[[r-q_n(z)]\left[1+\frac{a_m}{a_n}\mathrm{e}^{[c_1+c_2][z-s]}\right] + [r-q_m(z)]\left[1-\mathrm{e}^{[c_1+c_2][z-s]}\right]\right]\mathrm{d}z\Bigg\}.$$

The change in the future path of the water table in cell n resulting from a change in the water level in time t is therefore

$$\frac{\partial}{\partial x_n(t)}\left[\int_t^T x_n(s)\mathrm{d}s\right] = \int_t^T \left[\frac{a_n + a_m\mathrm{e}^{[c_1+c_2][t-s]}}{a_1+a_2}\right]\mathrm{d}s. \tag{21}$$

From Eq. (12), the price of each unit of water table height, $x_n(s)$, at time s is $\beta_n(s)$, and the price associated with the terminal height x_n^T at time T is β_n^T. The shadow price, or present discounted value (at time t) of the stream of losses incurred from a marginal drop in the water table at time t, is then the right hand side of Eq. (18). To convert the shadow value from a marginal change in water table height, x, to a marginal change in volume, q, it is necessary to divide by the area, a_n, thus obtaining the right hand side of Eq. (14).

For an isolated aquifer ($c_{mn} = 0$), the term $[a_n + a_m]\mathrm{e}^{[c_1+c_2][t-s]}/[a_1+a_2]$ reduces to unity, i.e., the full impact of extraction is on x_n. For the bathtub case ($c_{mn} \to \infty$), it reduces to $a_n/[a_1+a_2]$; the impact is proportional to the user's relative share of the aquifer.

Having characterized the unique Markov-perfect equilibrium induced by a linear mechanism, $\phi_n^L(\cdot)$, we now show that this type of mechanism can induce an arbitrary feasible and continuously differentiable extraction path over $[0, T]$.

More formally, let $\{\hat{q}_n(t) : t \in [0, T]\}$, be any continuously differentiable feasible extraction path satisfying $\mathrm{d}h_n(\hat{q}_n(t))/\mathrm{d}q_n < \infty$, and $q_n^{\boldsymbol{\phi}^L}(t)$ be the unique Markov-perfect equilibrium in open-loop strategies that is induced by linear-state mechanism $\boldsymbol{\phi}^L$. If $q_n^{\boldsymbol{\phi}^L}(t)$ is everywhere interior, we can show that there exists a unique mechanism such that $q_n^{\boldsymbol{\phi}^L}(t) = \hat{q}_n(t)$ for all $t \in [0, T]$ and $n = 1, 2$.

To this end, since $h_n(\cdot)$ is strictly concave, it is sufficient to show that for any $\hat{q}_n(t)$, terms $\beta_n(t)$ and β_n^T of mechanism ϕ_n^L can be chosen such that, for all $t \in [0, T]$,

$$\frac{\mathrm{d}h_n(\hat{q}_n(t))}{\mathrm{d}q_n} = \frac{\int_t^T \beta_n(s)\mathrm{e}^{\delta[t-s]} \frac{a_n + a_m \mathrm{e}^{[c_1+c_2][t-s]}}{a_1+a_2} \mathrm{d}s + \beta_n^T \mathrm{e}^{\delta[t-T]} \frac{a_n + a_m \mathrm{e}^{[c_1+c_2][t-T]}}{a_1+a_2}}{a_n}. \tag{22}$$

Set

$$\beta_n^T = a_n \mathrm{d}h_n(\hat{q}_n(T))/\mathrm{d}q_n, \tag{23}$$

ensuring that Eq. (22) is satisfied for $t = T$. Eq. (22) becomes

$$\int_t^T \beta_n(s)\mathrm{e}^{\delta[t-s]} \frac{a_n + a_m \mathrm{e}^{[c_1+c_2][t-s]}}{a_1 + a_2} \mathrm{d}s = a_n \frac{\mathrm{d}h_n(\hat{q}_n(t))}{\mathrm{d}q_n} - \beta_n^T \mathrm{e}^{\delta[t-T]} \frac{a_n + a_m \mathrm{e}^{[c_1+c_2][t-T]}}{a_1 + a_2}. \tag{24}$$

Performing the change of variable $z = T - t$, we have

$$\begin{aligned} -\int_0^z \beta_n(T-s)\mathrm{e}^{\delta[s-z]} \frac{a_n + a_m \mathrm{e}^{[c_1+c_2][s-z]}}{a_1 + a_2} \mathrm{d}s \quad &= \quad a_n \frac{\mathrm{d}h_n(\hat{q}_n(T-z))}{\mathrm{d}q_n} - \\ &\qquad \beta_n^T \mathrm{e}^{-\delta z} \frac{a_n + a_m \mathrm{e}^{-[c_1+c_2]z}}{a_1 + a_2}. \end{aligned} \tag{25}$$

Eq. (25) is a linear Volterra equation of the first kind with a kernel containing exponential functions (the general solution is derived in Polyanin and Manzhirov, 2008, p. 17). For our case, letting

$$g(z) = -a_n \frac{\mathrm{d}h_n(\hat{q}_n(T-z))}{\mathrm{d}q_n} + \beta_n^T \mathrm{e}^{-\delta z} \frac{a_n + a_m \mathrm{e}^{-[c_1+c_2]z}}{a_1 + a_2}, \tag{26}$$

the solution to (25) is given by

$$\beta_n(T-z) \quad = \quad \mathrm{e}^{-\delta z} \frac{\mathrm{d}}{\mathrm{d}z} \left\{ \mathrm{e}^{-\frac{a_n[c_1+c_2]z}{a_1+a_2}} \int_0^z \frac{\mathrm{d}}{\mathrm{d}s} \left[g(s) \mathrm{e}^{[\delta+c_1+c_2]s} \right] \mathrm{e}^{\frac{a_m[c_1+c_2]s}{a_1+a_2}} \mathrm{d}s \right\}. \tag{27}$$

The differentiability of $h_n(\cdot)$ and $\hat{q}_n(\cdot)$ ensures that Eq. (27) is well-defined. Repeating this argument for user m and collecting the $\beta_n(\cdot), \beta_m(\cdot)$ functions and β_n^T, β_m^T constants establishes the desired result.

To summarize, first, the linear mechanism causes each user to attach a shadow price to the water table level of his cell at each moment in time (which otherwise would have been zero). Second, the regulator can parametrize a linear mechanism to induce any feasible extraction path (that is continuously differentiable and results in finite first derivatives of the restricted profit functions) in Markov-perfect equilibrium. Finally, if the socially optimal path satisfies these conditions then the regulator can induce it with some linear mechanism. Moreover, inspection of Eq. (12) shows that in equilibrium no transfers take place. That is to say, the *threat* of linear transfers is sufficient to induce socially optimal behavior.[16]

In practical terms, Theorem 1 suggests that eliminating the simplifying single-cell bathtub assumption may not result in a hopelessly complex regulatory structure. On the contrary, even if an arbitrary number of users act strategically in a manner consistent with Markov perfection, and the regulator cannot observe their behavior directly, and they occupy an arbitrary number of cells with differing hydrology, the socially optimal extraction rate can be induced by a dynamic linear price path for the difference between a user's observed water table depth and the socially optimal one.

It is important to note that the *linearity* of the price mechanism does not depend on the underlying hydrology, but the *value* of the price itself does. In particular, the price path

[16]We assume that the users do not doubt the regulator's capacity to follow through on this threat.

for each user (defined for the two-user case in Eq. (27)) depends upon the hydrological connectivity between himself and all other users. Thus, the mechanism would not be easy to calculate, but once the prices were derived it would be relatively simple to implement.

This simplicity rests on the fact that together the two restrictions of inexhaustibility and no internalized extraction cost imply that an individual user need not take other user behavior into account when determining his equilibrium strategy. For the two-player case, this result can be most clearly seen in Eqs. (14) and (18). By Eq. (14), each user sets his marginal benefit of extraction equal to λ_{nn}. Eq. (18) indicates that λ_{nn} is a function of hydrological characteristics and the regulator's price instruments $\beta_n(t)$. In sum, each user's equilibrium strategy is unaffected by that of his neighbors. Eqs. (23), (27), and (12) reflect this fact, implying that the instrument's optimal values depend only upon user n's marginal profit from a unit of extraction and hydrological characteristics, *not* other users' state variables. In contrast, as shown in Section 4.2 it is likely to be much more difficult to implement a mechanism that induces the social optimum for aquifers that do not satisfy either the inexhaustibility no internal extraction cost assumptions.

While it is obvious that the social welfare induced by the optimal multi-cell mechanism can be no lower than that induced if a regulator were to erroneously believe that an aquifer were a bathtub, the theoretical structure of the linear mechanism does not provide any unambiguous information regarding the direction or magnitude of the distortions in prices or extraction rates that a mistaken regulator might cause; these effects may vary by the specific characteristics of the hydrology and users. In section 5 we conduct a numerical simulation to illustrate how a regulator might clarify these issues.

4.2 Resource may be exhausted by T and/or some internal extraction cost

The Markov-perfection of the equilibrium induced by the linear mechanism rests on two critical assumptions. First, that the users bear no extraction cost ($\mu = 0$), and second that the aquifer cannot be exhausted before time T. Although these conditions may hold approximately for some aquifers in parts of countries such as India that do not charge marginal electricity rates, they are unlikely to apply in general.

In this section, we analyze the effects of relaxing these two assumptions. It is clear that open loop strategies involving extraction strictly greater than recharge in a time period in which exhaustion is possible cannot form a Markov-perfect equilibrium (since such extraction would not be feasible in all possible subgames). One can, however, utilize the Hamilton-Jacobi-Bellman dynamic programming approach to derive a mechanism that induces the social optimum in Markov perfect equilibrium. We derive the general case covering any degree of internalization of extraction costs (i.e., $\mu \in [0,1]$), thus addressing the positive extraction case in passing.

For the sake of algebraic tractability, we explicitly employ the quadratic restricted profit function, specifying it as[17]

$$h_n(q_n) = -\frac{\alpha_1}{2}q_n^2 + \alpha_2 q_n, \quad \alpha_1,\ \alpha_2 > 0. \tag{28}$$

As before, the regulator's goal is to solve Eq. (4).[18] Similarly, given a general mechanism $\boldsymbol{\phi}$, the users play the differential game described in Eq. (10). The key differences with the previous section are that for this game $\mu \geq 0$ and we explicitly account for the state non-negativity constraint $\mathbf{x} \geq \mathbf{0}$.

Before deriving equilibrium behavior and characterizing the optimal mechanism, we show that the social optimum itself admits a representation amenable to the Hamilton-Jacobi-Bellman approach. Specifically, we need to show that socially optimal extraction functions are polynomial in $\mathbf{x}$.

Let $V^{SO}(\mathbf{x}, t)$ denote the socially optimal value function, defined by the solution to (4)

[17] To simplify notation, here we assign each user an identical restricted profit function. In the numerical simulations in the next section we relax this assumption.

[18] For convenience, we assume that $D^T \equiv 0$ and that (while it is feasible) it is never *socially optimal* to exhaust the aquifer, regardless of initial water table levels.

at time t. The corresponding Hamilton-Jacobi-Bellman equation is

$$\begin{aligned}
\delta V^{SO}(\mathbf{x},t) - \frac{\partial}{\partial t} V^{SO}(\mathbf{x},t) &= \max_{\mathbf{q}\in[\mathbf{0},\bar{\mathbf{q}}]} \left\{ \sum_{n=1}^{N} -\frac{\alpha_1}{2} q_n^2 + \alpha_2 q_n - d(q_n(t)), x_n(t)) \right. \\
&\quad \left. + \sum_{n=1}^{N} \frac{\frac{\partial}{\partial x_n} V^{SO}(\mathbf{x},t)}{a_n} \left[r - q_n + \sum_{m=1}^{N} c_{mn}[x_m - x_n] \right] \right\}. \\
V^{SO}(\mathbf{x},T) &= 0 \qquad (29)
\end{aligned}$$

The following Lemma establishes that the extraction strategies solving this problem, $q_n^{SO}(\mathbf{x},t)$, are in fact polynomial (specifically, linear) in $\mathbf{x}$.

Lemma 2 *Suppose Eq. (28) holds and that $D^T \equiv 0$ and let $\mathbf{q}^{SO}(\mathbf{x},t)$ denote the solution to (4). There exist some time-dependent functions $\{k_n(t)\}_{n=1}^N$ and $\{k_{mn}(t)\}_{m,n=1}^N$ such that*

$$q_n^{SO}(\mathbf{x},t) = \sum_{m=1}^{N} k_{nm}(t) x_m + k_n(t), \quad n = 1,2,...,N. \qquad (30)$$

We now derive a mechanism that induces socially optimal extraction in Markov-perfect equilibrium. Let $V_n(\mathbf{x},t)$ denote the value function of user n at state $\mathbf{x}$ and time t. Given a mechanism $\boldsymbol{\phi}$, the Hamilton-Jacobi-Bellman equation for user n, assuming that other users $m \neq n$ use their socially optimal strategies $q_m^{SO}(\mathbf{x},t)$ given by Eq. (55) is

$$\begin{aligned}
\delta V_n(\mathbf{x},t) - \frac{\partial}{\partial t} V_n(\mathbf{x},t) &= \max_{q_n\in[0,\bar{q}]} \left\{ \frac{-\alpha_1 q_n^2}{2} + \alpha_2 q_n - \mu q_n[\omega_1 + \omega_2[\bar{x} - x_n]] + \phi_n(\mathbf{x},t) \right. \qquad (31) \\
&\quad + \frac{\frac{\partial}{\partial x_n} V_n(\mathbf{x},t)}{a_n} \left[r - q_n + \sum_{m=1}^{N} c_{mn}[x_m - x_n] \right] \\
&\quad \left. + \sum_{m\neq n} \frac{\frac{\partial}{\partial x_m} V_n(\mathbf{x},t)}{a_m} \left[r - q_m^{SO}(\mathbf{x},t) + \sum_{\ell=1}^{N} c_{\ell m}[x_\ell - x_m] \right] \right\}, t < T; \\
V_n(\mathbf{x},T) &= \phi_n(\mathbf{x},T). \qquad (32)
\end{aligned}$$

The following theorem shows that the family of second degree polynomial mechanisms induces the socially optimal extraction path in Markov-perfect equilibrium. In particular, consider mechanisms $\boldsymbol{\phi}^P(\mathbf{x}(t),t) = \left(\phi_1^P(\mathbf{x},t), \phi_2^P(\mathbf{x},t), ..., \phi_N^P(\mathbf{x},t)\right)'$, where:

$$\phi_n^P(\mathbf{x},t) = \begin{cases} \left[\mathbf{x} - \mathbf{x^{SO}}(t)\right]'\mathbf{f_n(t)} + \frac{1}{2}[\mathbf{x} - \mathbf{x^{SO}}(t)]'\mathbf{F_n(t)}[\mathbf{x} - \mathbf{x^{SO}}(t)], & t < T. \\ \left[\mathbf{x} - \mathbf{x^{SO}}(T)\right]'\mathbf{f_n^T} + \frac{1}{2}[\mathbf{x} - \mathbf{x^{SO}}(T)]'\mathbf{F_n^T}[\mathbf{x} - \mathbf{x^{SO}}(T)], & t = T \end{cases} \tag{33}$$

in which (a) $\mathbf{f_n(t)}$ and $\mathbf{f_n^T}$are N-dimensional vectors and $\mathbf{F_n(t)}$ and $\mathbf{F_n^T(t)}$ are $N \times N$-dimensional matrices. The structure of mechanism $\boldsymbol{\phi}^P(\mathbf{x}(t),t)$ is similar to that of policies designed for dynamic nonpoint-source pollution control (Xepapadeas, 1992; Athanassoglou, 2010).

Theorem 2 *There exists a (possibly non-unique) second-degree polynomial mechanism satisfying (33) that induces the socially optimal extraction strategy* $\mathbf{q}^{SO}(\mathbf{x},t)$ *in Markov-perfect equilibrium.*

In contrast to the simple linear transfer obtained in the previous section, the optimal mechanism is a polynomial function of the water table level in a user's cell and the water tables of *all* other cells.

Note that the optimal mechanism whose existence is proved in Theorem 2 is a function of the state variables of all users. This structure reflects the fact that relaxing the restriction of either inexhaustibility or no internalization of extraction costs creates a strategic interaction amongst the users that was not present in the case analyzed in Section 4.1. For cases in which the users bear some extraction cost, their equilibrium strategies obviously depend on the strategies of other users; user n's profit depends on the value of his state variable, which in turn depends on the value of the user m's state variable. Thus, the optimal mechanism characterized by Eq. (33) must also be a function of other users' state variables in order to incorporate this effect.

A similar situation arises if the non-negativity constraint may bind, even if all other extraction costs are external. Effectively, a binding non-negativity constraint increases the marginal extraction cost in a non-continuous way from zero to infinity for all extraction above the natural recharge rate. As in the case with extraction costs partly internalized, the possibility of a binding non-negativity constraint ensures that user n's equilibrium extraction path is a function of user m's state variable even absent government intervention. The optimal policy takes this interaction into account by making transfers dependent on all state variables.

5 Numerical Simulations

In this section, we numerically simulate the differential game (10) for two agricultural users in a rural setting in semi–arid tropical India. In these regions, agricultural production was traditionally constrained by precipitation variations during the wet monsoon season. The advent of inexpensive pump technology in the 1970s coupled with subsidized electricity now allows year-round production (Shah, 2008; Reddy, 2005).

Table 1 lists the parameters used in the simulation.[19] We calculate monetary units in 2005 U.S. dollars, using the average annual exchange rate of 44 Rupees per dollar. Farmers are adjacent landholders with one hectare plots. They share a watershed that receives no recharge through lateral subsurface inflows over the boundary. As in the theoretical model, a hydraulic connection between the adjacent landholdings allows water to flow across this interface depending on the individual water table elevations. We assume homogeneous and isotropic aquifer properties and choose parameter values representative of subsurface properties of weathered crystalline rock found in large parts of peninsular India. We suppose constant characteristic values for the hydraulic transmissivity c. For both farmers, initial drawdown levels are at 20 meters below ground surface.

The agro-economic parameters are representative of small landholders growing paddy rice in two seasons per year. Each farmer pumps water from one borehole located on his plot. We specify restricted profit (net returns to land) as a quadratic function of water input for users $n = 1, 2$:[20]

$$h_n(q_n) = \theta_n \left[\alpha_1 q_n + \alpha_2 q_n^2\right]. \tag{34}$$

[19] See Raj (2004) for data on climate and groundwater, Shiferaw et al. (2008) for crop and agricultural production specific data, and World Bank (2001) plus references therein for energy data.

[20] This specification implicitly assumes rainfed agricultural production is infeasible.

Parameter	Description	Value	Unit
A	landholding size	1	ha
S	effective porosity	0.01	n.a.
c	seasonal transmissivity	6	m^2
$\bar{x} - x_n(t)$	drawdown	20	m
r	seasonal recharge	0.04	m
δ	discount rate	0.05	n.a.
θ_1	productivity scaling factor	1	n.a.
θ_2	productivity scaling factor	0.9	n.a.
α_1	technological parameter	0.08	US$ / m^3
α_2	technological parameter	-6.2×10^{-6}	US$ / m^6
ω_1	energy cost parameter	2.3×10^{-3}	US$ / m^3
ω_2	energy cost parameter	2.3×10^{-4}	US$ / m^4

Table 1: Simulation parameter values

Panel a of Figure 1 illustrates restricted profit functions for both farmers. Farmer 1 is more technically efficient in the sense that he can attain any feasible profit using less water than farmer 2.

Social costs reflect typical expenses for the state related to provision of rural energy and are presumed to be the same for both farmers. We continue to assume the standard energy cost function specified in Eq. (3). We set the terminal time cost to $D^T(\mathbf{x}(T)) = 0$.

5.1 Socially optimal mechanism

Initial simulation results are shown in Figure 1.[21] Socially optimal (SO) pumping rates decline over time (Panel b in Figure 1). Privately optimal (PO) pumping rates, representing the outcome of the unregulated status quo, are constant throughout the optimization period since extraction costs are not internalized by the users.[22] Panel c indicates the mechanism price path necessary to induce socially optimal extraction; terminal prices are $24.50 and

[21]For all computations we use Matlab (2010b). We solve the system of differential equations as a non-linear programming problem using the control vector parameterization concept by utilizing the DOTcvp toolbox (Hirmajer et al., 2008). We utilize the fmincon function as part of Matlab's optimization toolbox in combination with DOTcvp to obtain the socially optimal solution, with a seasonal discrete time-step over 10 years.

[22]The privately optimal solution is characterized by Eq. (14) with $\lambda_{nn} = 0$, i.e., users set their pumping rates such that the marginal profit from extraction is equal to zero in each moment in time.

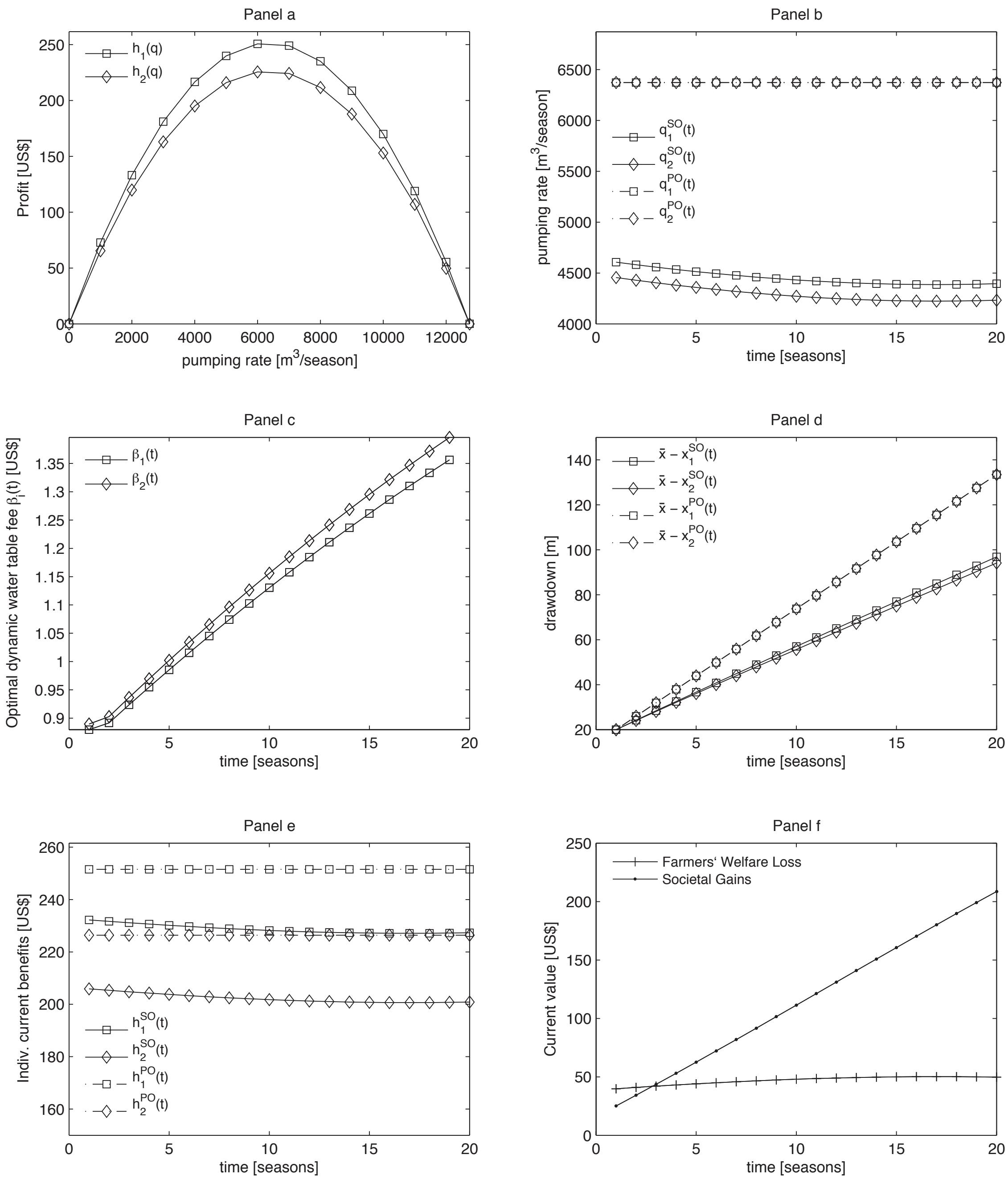

Figure 1: Simulation results for socially optimal (SO) and privately optimal (PO) extraction rates.

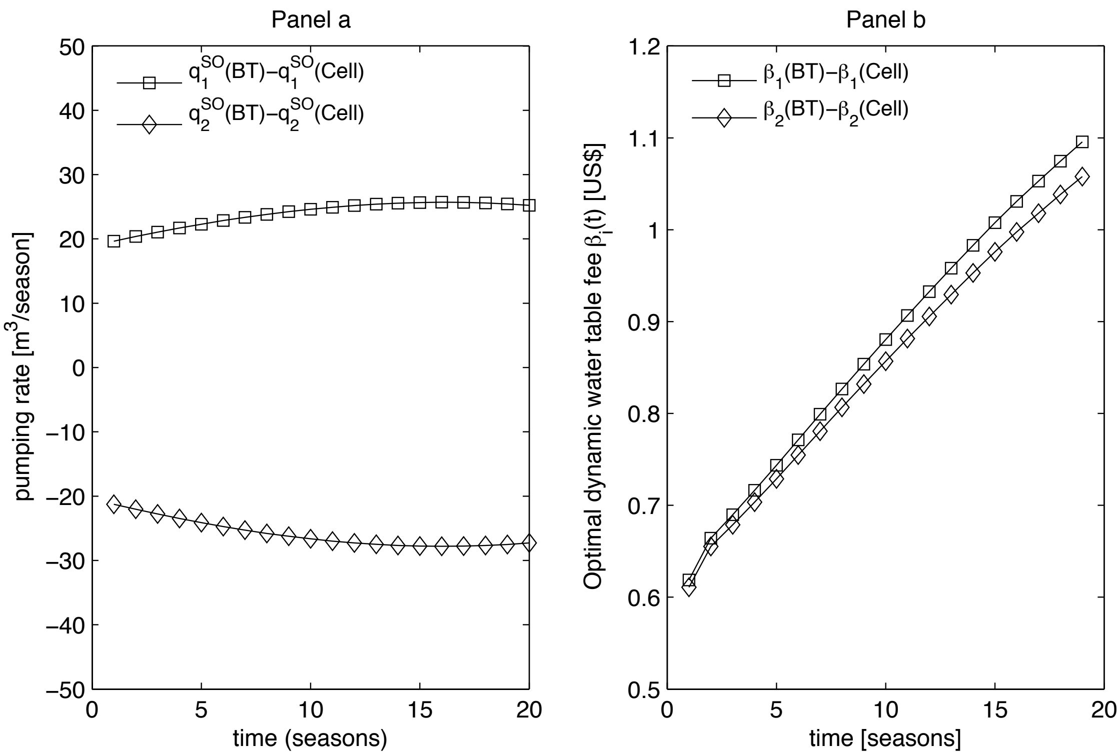

Figure 2: Difference in optimal extraction and mechanism prices between bathtub (BT) and multi-cell (Cell) assumption

$23.86 for users 1 and 2, respectively. Panel e illustrates the impact of the conservationist optimal policy on the resource supply. Note that since socially optimal extraction is lower for user 2, drawdown is lower and water flows from cell 2 to cell 1, whereas in the privately optimal equilibrium there are no transboundary flows since drawdown is equal.

The lower extraction rate hurts farmer profit relative to the private optimum as indicated in panel e. Of course, as shown in panel f, the net welfare result is positive; in general the societal savings in extraction costs outweigh the loss in farm profit. The optimal policy does have a negative net impact in the first few time periods, however.

5.2 Role of spatial and economic complexity

In this section we discuss welfare loss from making two kinds of mistakes in implementing the optimal mechanism: (i) incorrectly assuming that the underlying aquifer is a bathtub, or; (ii) incorrectly assuming that users are homogeneous.

Under assumption (i), the regulator solves for the socially optimal extraction path assuming incorrectly that the aquifer has infinite transmissivity. In particular, she solves the optimal control problem given by Expression (4) with state dynamics given by Lemma 1. Then, she plugs the derived extraction path into Eq. (27), assuming that $c_1 = c_2 = \infty$, to obtain the mechanism charges. Implementation of this mechanism results in the equilibrium characterized by Eqs. (14), (18), and (19) induced by the incorrect prices.

Unsurprisingly, Figure 3 shows that aquifer dynamics have a major effect on optimal policy when transmissivity is low, a feature commonly found in hard rock or well consolidated sedimentary formations. The graph depicts percentage welfare loss (with regard to the social optimum) as a function of actual field transmissivity values c. The range over which c is varied (0.3 to 2×10^4 m^2/season) corresponds to field situations as reported in Raj (2004). The negative impact increases the less the aquifer resembles a bathtub in reality. As transmissivity increases, the bathtub assumption results in less welfare loss and eventually becomes innocuous.

We now address the issue of whether an incorrect bathtub assumption would lead a regulator to err on the side of over-conserving the resource.[23] It seems plausible that it would be the case since the effect of one user's extraction on other users' costs is maximized in a single cell relative to an aquifer with multiple cells with limited connectivity. Nonetheless, we provide a simple counter-example illustrating why this conjecture is not necessarily true.

Panel a of Figure 2 compares the optimal extraction paths under the two connectivity scenarios. For user 2, the conjecture is correct; the bathtub extraction path is more conservative. For user 1, however, the opposite is true. Essentially, by believing that the aquifer is a bathtub, the regulator underestimates the true cost to user 1 of accessing the water in cell 2, thus over-prescribing his extraction rate relative to the actual optimum. Thus, in aquifers with heterogeneous users, an erroneous bathtub assumption can lead to over-extraction for

[23]We thank an anonymous referee for raising this question.

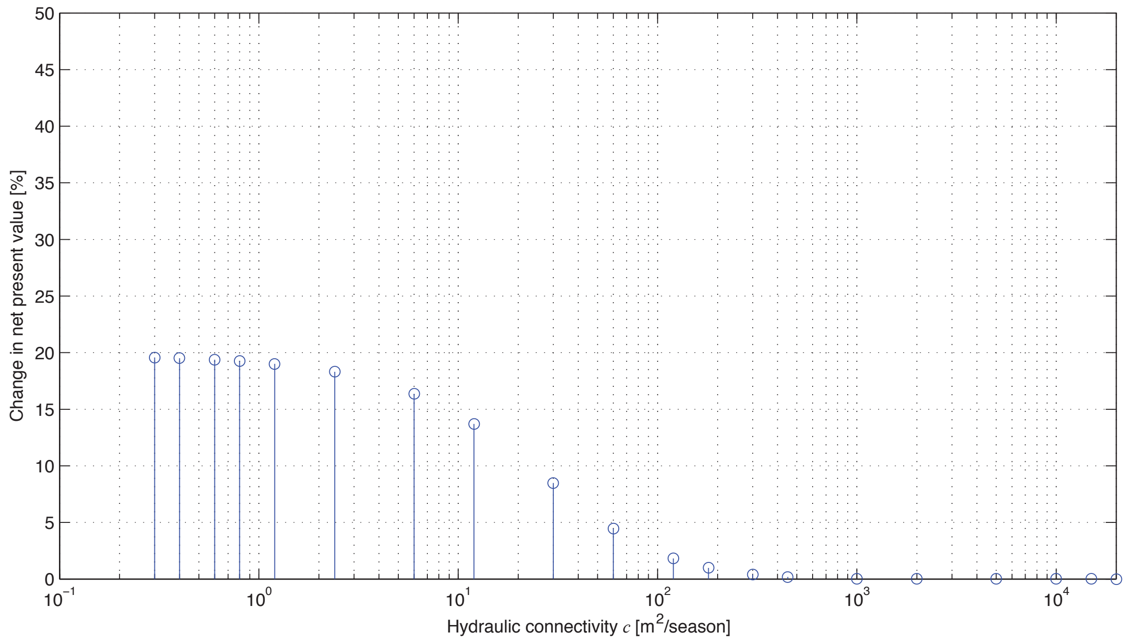

Figure 3: Percent social welfare loss from incorrect "bathtub" assumption.

some users.

This reasoning readily extends to cases in which users are not evenly distributed across an aquifer. As a simple example, consider a two-cell aquifer with only one user in the first cell. If the regulator mistakenly believed that the aquifer was a bathtub she would under-estimate the cost to the user of accessing the water in the second cell, thus prescribing excessive extraction relative to the true optimum.

Panel b indicates the effect of an incorrect bathtub assumption on mechanism prices. Until the final period the charges are higher under a bathtub for both users. In the last period (not shown on graph), however, the bathtub prices are lower for user 1 are lower ($24.18), for a net effect of increasing extraction. For user 2, the last period charges are higher under the bathtub assumption ($24.16).

Regarding assumption (ii), economic heterogeneity is defined as the ratio between the value of the two profit functions in Eq. (34). In our calculations, we take θ_1 as given and vary θ_2 from 0.1 to 1. The regulator's mistake is now the following. First, she solves for the socially optimal extraction path assuming incorrectly that the two users have identical

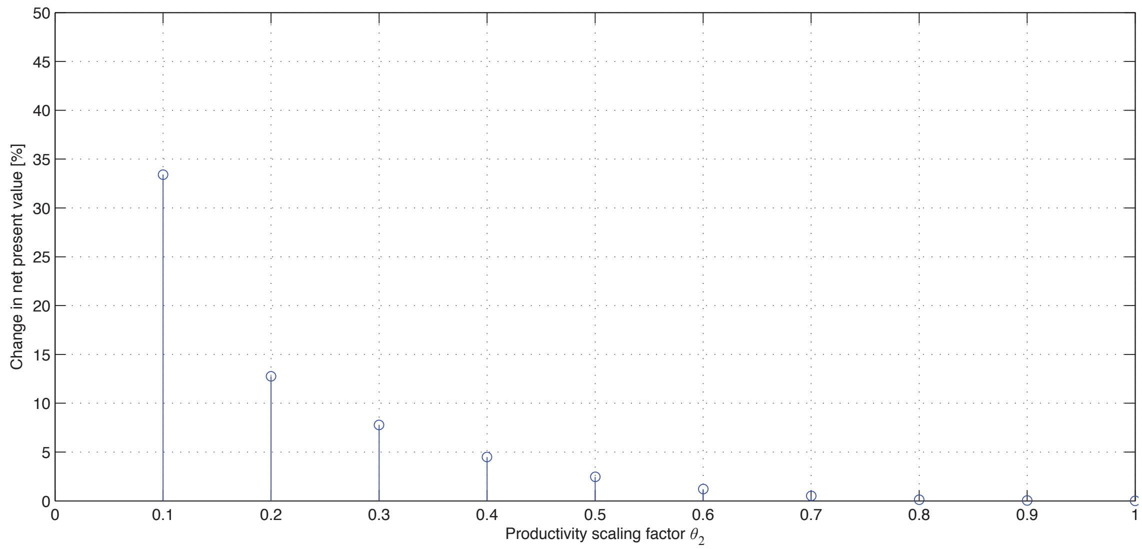

Figure 4: Percent social welfare loss from incorrect farm homogeneity assumption.

profit functions. That is, she solves the optimal control problem given by Expression (4) supposing that $\theta_2 = 1$ in the objective function. Then, she plugs the derived extraction path into Eq. (27) to obtain the mechanism charges. Implementation of this mechanism results in the equilibrium characterized by Eqs. (14), (18), and (19) induced by the prices calculated based upon the incorrect θ_2.

Figure 4 depicts percentage welfare loss (with regard to the social optimum) as a function of actual heterogeneity. The simulation suggests that adverse welfare impacts increase with user heterogeneity, potentially reaching high levels.

6 Conclusion

Previous literature on strategic behavior among users of an aquifer has abstracted from the complicating factors of imperfect transmissivity in groundwater flows and user heterogeneity. This paper provides a step forward by presenting an analytical framework for dealing with both these issues. We divide aquifers into two classes. The first class cannot physically be exhausted in the finite period of interest and users do not bear any costs of extraction. The second class can be exhausted and/or has users that bear some extraction cost.

Unlike previous work, we present a mechanism for inducing a socially optimal extraction path in Markov perfect equilibrium. In spite of the underlying complexity of the system, and the government's inability to monitor individual user extraction, the optimal mechanism for the first class of aquifers is quite simple: a dynamic linear price based upon the height of a user's (potentially shared) local water table. This result holds irrespective of the aquifer's physical characteristics (number of cells, transmissivity) as well as the number and spatial configuration of extracting users. The optimal mechanism we derive for the second class of aquifers is more complicated. It is a polynomial function of not only an individual user's local water table, but the water table for all users.

The intuition behind the qualitative difference between the two mechanisms is straightforward. In the first case, an individual user's strategy is independent of the actions of his neighbors. This independence is due to the fact that the user does not bear extraction costs. Thus, his welfare is independent of all state variables. In the second case, this is not true. A user's welfare is dependent on his neighbor's actions through their effects on his state variable. The state variable affects his welfare directly if he internalizes some extraction costs. It also affects his welfare if he bears no explicit extraction costs, but the aquifer is exhaustible. In that case, if his cell becomes exhausted the marginal extraction cost implicitly rises from zero to infinity.

Although the theoretical results indicate how to design policies that take into account a heterogeneous aquifer, they cannot indicate the potential amount of gains relative to policies prescribed by an overly simple alternative model. Fortunately, our multi-cell approach has the additional advantage of lending itself well to numerical modeling methods commonly used in the hydrology literature. We conduct a number of simulations using aquifer characteristics in India that suggest that there may be substantial efficiency losses from models that erroneously either assume an incorrect bathtub hydrology, or that all users are identical.

We also show that, in addition to being inefficient, an incorrect bathtub model can lead to over-extraction of the resource. This counter-intuitive result is due to the fact that a bathtub model in effect underestimates the cost of accessing water in other cells, while overestimating the external cost imposed on neighboring users. Over-extraction can occur if the first effect dominates.

Apart from the economic efficiency issues analyzed here, our model has potential applications in distributional analysis. The importance of accounting for spatial complexity and heterogeneous users increases further if equity is considered. In developing countries, agriculture is often characterized by large land holdings of relatively wealthy owners and small tracts worked by poorer households. Distributional welfare analysis in such settings requires a model of water extraction that approximates actual water table dynamics while allowing for strategic behavior among heterogeneous users. Our model provides an analytical framework that can use such characteristics in numerical simulations to evaluate different policy scenarios.

Like any modeling exercise, the approach developed here relies on a set of simplifying assumptions that suggest both caveats and potentially fruitful courses of future research. Although our policy does not rely on monitoring individual extraction rates of users sharing a micro-watershed, it does assume costless continuous monitoring of deterministic water table levels. An interesting avenue of research would be to determine the welfare implications arising from imperfect monitoring in both space and time (e.g., if the regulator can only check water table levels at a subset of locations at discrete intervals) and accounting for stochastic flows.

Appendix

Proof of Lemma 1. (i) Letting $\lim_{c\to 0} q_n(t;0) = q_n(t;0)$, the dynamics of an unconnected aquifer are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\lim_{c\to 0} x_n(t;c)\right] = \lim_{c\to 0}\frac{\mathrm{d}}{\mathrm{d}t}\left[x_n(t;c)\right] = \frac{r - q_n(t;0)}{a_n}, \text{ for all } t \in [0,T]. \tag{35}$$

Taking limits as $c \to 0$, Eqs. (1) and (2) arrive at this expression.

(ii) Letting $\lim_{c\to\infty} q_n(t;c) = q_n(t;\infty)$, the dynamics of a single-cell aquifer are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\lim_{c\to\infty} x_n(t;c)\right] = \lim_{c\to\infty}\frac{\mathrm{d}}{\mathrm{d}t}\left[x_n(t;c)\right] = \frac{2r - q_1(t;\infty) - q_2(t;\infty)}{a_1+a_2}, \text{ for all } t\in[0,T]. \tag{36}$$

Consider the water table equations $x_n(t,c)$ given by Eq. (2). Since the function $q_n(t,c)$ is bounded, the Bounded Convergence Theorem (see Rudin, 1976) implies that the integral of the limit is equal to the limit of the integral. Taking the limit as $c\to\infty$ in Eq. (2) yields:

$$\lim_{c\to\infty} x_n(t;c) = \frac{a_n x_n(0) + a_m x_m(0) + 2rt - \int_0^t q_1(s;\infty) + q_2(s;\infty)\mathrm{d}s}{a_1+a_2}. \tag{37}$$

Differentiating with respect to t,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\lim_{c\to\infty} x_n(t;c)\right] = \frac{2r - q_1(t;\infty) - q_2(t;\infty)}{a_1+a_2}. \tag{38}$$

Substituting $x_n(t;c)$ and $x_m(t;c)$ from Eq. (2) into user n's dynamics yields:

$$\begin{aligned}
a_n\dot{x}_n(t;c) &= r - q_n(t;c) + \tfrac{c}{a_1+a_2}\Big\{\mathrm{e}^{-[c_1+c_2]t}\left[x_m(0)a_m\left[1+\tfrac{a_n}{a_m}\right] - x_n(0)a_n\left[1+\tfrac{a_m}{a_n}\right]\right] + \\
&\quad \int_0^t \mathrm{e}^{[c_1+c_2][s-t]}\left[r\left[\tfrac{a_n}{a_m}-\tfrac{a_m}{a_n}\right] + q_n(s;c)\left[1+\tfrac{a_m}{a_n}\right] - q_m(s;c)\left[1+\tfrac{a_n}{a_m}\right]\right]\mathrm{d}s\Big\} \\
&= r\left[\tfrac{2a_n}{a_1+a_2} - \tfrac{a_n^2-a_m^2}{[a_1+a_2]^2}\mathrm{e}^{-[c_1+c_2]t}\right] \\
&\quad -q_n(t;c) + \tfrac{c}{a_1+a_2}\Big\{\mathrm{e}^{-[c_1+c_2]t}\left[x_m(0)a_m\left[1+\tfrac{a_n}{a_m}\right] - x_n(0)a_n\left[1+\tfrac{a_m}{a_n}\right]\right] \\
&\quad + \int_0^t \mathrm{e}^{[c_1+c_2][s-t]}\left[q_n(s;c)\left[1+\tfrac{a_m}{a_n}\right] - q_m(s;c)\left[1+\tfrac{a_n}{a_m}\right]\right]\mathrm{d}s\Big\} \\
&= r\left[\tfrac{2a_n}{a_1+a_2} - \tfrac{a_n^2-a_m^2}{[a_1+a_2]^2}\mathrm{e}^{-[c_1+c_2]t}\right] \\
&\quad -q_n(t;c) + \tfrac{c}{a_1+a_2}\mathrm{e}^{-[c_1+c_2]t}\left[x_m(0)a_m\left[1+\tfrac{a_n}{a_m}\right] - x_n(0)a_n\left[1+\tfrac{a_m}{a_n}\right]\right] \\
&\quad + \tfrac{a_m}{a_1+a_2}\left[q_n(t;c) - \mathrm{e}^{-[c_1+c_2]t}q_n(0;c) - \int_0^t \mathrm{e}^{[c_1+c_2][s-t]}\tfrac{\mathrm{d}}{\mathrm{d}s}[q_n(s;c)]\mathrm{d}s\right] \\
&\quad - \tfrac{a_n}{a_1+a_2}\left[q_m(t;c) - \mathrm{e}^{-[c_1+c_2]t}q_n(0;c) - \int_0^t \mathrm{e}^{[c_1+c_2][s-t]}\tfrac{\mathrm{d}}{\mathrm{d}s}[q_m(s;c)]\mathrm{d}s\right].
\end{aligned} \tag{39}$$

Taking the limit of this expression as $c\to\infty$ and applying the Bounded Convergence Theorem yields the desired result. ■

Proof of Theorem 1. We wish to find $\boldsymbol{\phi}^L$ such that

$$\frac{\mathrm{d}h_n(\hat{q}_n(t))}{\mathrm{d}q_n} = \frac{\lambda_{nn}^{\phi^L}(t)}{a_n}, \quad \text{for all } t \in [0,T]. \tag{40}$$

where the vector of co-state variables $\boldsymbol{\lambda_n^{\phi^L}} = \left(\lambda_{1n}^{\phi^L}(t), \lambda_{2n}^{\phi^L}(t), ..., \lambda_{Nn}^{\phi^L}(t)\right)'$ solves the following system of differential equations with terminal conditions:

$$\begin{aligned} &\dot{\boldsymbol{\lambda}}_{\boldsymbol{n}}^{\boldsymbol{\phi^L}} = \mathbf{A} \cdot \boldsymbol{\lambda}_n^{\phi^L}(t) + \mathbf{b_n(t)} \\ &\lambda_{nn}^{\phi^L}(T) = \beta_n^T, \;\; \lambda_{mn}^{\phi^L}(T) = 0 \;\text{ for all } m \neq n. \end{aligned} \tag{41}$$

Here $\mathbf{A} \in \Re^{N\times N}$ and $\mathbf{b}_n \in \Re^N$ are defined such that

$$A_{nn} = \delta + \frac{1}{a_n}\sum_{m=1}^{N} c_{mn}; \tag{42}$$

$$A_{mn} = -\frac{c_{mn}}{a_n}, \qquad m \neq n; \tag{43}$$

$$b_{nn}(t) = -\beta_n(t), \;\; \text{and} \;\; b_{mn} = 0, \;\; m \neq n. \tag{44}$$

A general solution for the system of linear differential equations given by Eqs. (41) can be found in Chapter 2.3.4 of Coddington and Carlson (1997):

$$\boldsymbol{\lambda}_n^{\phi}(t) = \boldsymbol{\Lambda}_n(t)\boldsymbol{\xi} + \boldsymbol{\Lambda}_n(t)\int_0^t \left[\boldsymbol{\Lambda}_n(s)\right]^{-1}\mathbf{b}_n(s)\mathrm{d}s, \;\; t \in [0,T], \tag{45}$$

where $\boldsymbol{\xi} \in \Re^n$ and $\boldsymbol{\Lambda}_n(t)$ is a basis for the solutions to the homogeneous counterpart of system (41). Performing the change of variable $z = T - t$, choosing $\boldsymbol{\Lambda}_n$ so that $\boldsymbol{\Lambda}_n(z) = \mathbf{I}$ at $z = 0$, and setting $\boldsymbol{\xi}$ to a vector $\boldsymbol{\xi}^{\beta_n^T}$ such that the transversality conditions in Eqs. (41) are satisfied,[24] yields the following unique solution of system (41)

$$\boldsymbol{\lambda}_n^{\phi^L}(z) = \boldsymbol{\Lambda}_n(z)\boldsymbol{\xi}^{\beta_n^T} - \boldsymbol{\Lambda}_n(z)\int_0^z \left[\boldsymbol{\Lambda}_n(s)\right]^{-1}\mathbf{b}_n(T-s)\mathrm{d}s, \;\; z \in [0,T]. \tag{46}$$

[24]Since the matrix $\boldsymbol{\Lambda}_n$ has full rank, $\boldsymbol{\xi}^{\beta_n^T}$ exists and is uniquely determined.

Denote row m of matrix $\mathbf{\Lambda}_n$ by $\mathbf{\Lambda}_{mn}$. The restriction of vector (46) to coordinate n obtains

$$\lambda_{nn}^{\phi^L}(z) = \mathbf{\Lambda}_{nn}(z)\boldsymbol{\xi}^{\beta_n^T} - \int_0^z \Big[\mathbf{\Lambda}_n(z)\big[\mathbf{\Lambda}_n(s)\big]^{-1}\Big]_{nn}\beta_n(T-s)\mathrm{d}s, \quad t \in [0,T], \quad z \in [0,T]. \tag{47}$$

Using Eq. (47), we obtain the following Volterra integral equation of the first kind

$$\begin{aligned} -a_n\left[\frac{\mathrm{d}h_n(\hat{q}_n(T-z))}{\mathrm{d}q_n}\right] + \mathbf{\Lambda}_{nn}(z)\boldsymbol{\xi}^{\beta_n^T} \quad &= \\ &\int_0^z \Big[\mathbf{\Lambda}_n(z)\big[\mathbf{\Lambda}_n(s)\big]^{-1}\Big]_{nn}\beta_n(T-s)\mathrm{d}s, \quad \text{for all } z \in [0,T]. \end{aligned} \tag{48}$$

We set β_n^T so that Eq. (49) is satisfied for $z = 0$. The integral equation's kernel

$$\Theta(z,s) = \Big[\mathbf{\Lambda}_n(z)\big[\mathbf{\Lambda}_n(s)\big]^{-1}\Big]_{nn} \tag{49}$$

is such that $\Theta(z,z) = 1$. This fact, in combination with our differentiability assumptions, implies that Eq. (49) may be reduced to the following equivalent Volterra integral equation of the second kind

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}\left[-a_n\left[\frac{\mathrm{d}h_n(\hat{q}_n(T-z))}{\mathrm{d}q_n}\right] + \mathbf{\Lambda}_{nn}(z)\boldsymbol{\xi}^{\beta_n^T}\right] \quad &= \\ &\beta_n(T-z) + \int_0^z \frac{\mathrm{d}}{\mathrm{d}z}\Theta(z,s)\beta_n(T-s)\mathrm{d}s, \quad z \in [0,T]. \end{aligned} \tag{50}$$

Our continuity and differentiability assumptions ensure that Theorem 2.1.1 in Burton (2005) applies and integral equation (50) has a unique solution. Repeating the argument for all users establishes the result. ■

Proof of Lemma 2. Assuming no corner solutions for the social optimum, first-order conditions on q_n imply:

$$q_n^{SO}(\mathbf{x},t) = \frac{1}{\alpha_1}\left[\alpha_2 - \omega_1 - \omega_2\big[\overline{x} - x_n\big] - \frac{\frac{\partial}{\partial x_n}V^{SO}(\mathbf{x},t)}{a_n}\right]. \tag{51}$$

Substituting (51) into the Hamilton-Jacobi-Bellman equation (29) yields, after some algebraic manipulation

$$\delta V^{SO}(\mathbf{x},t) - \frac{\partial}{\partial t}V^{SO}(\mathbf{x},t) = \sum_{n=1}^{N} \frac{\left(\alpha_2 - \omega_1 - \omega_2\left[\overline{x} - x_n\right] - \frac{\frac{\partial}{\partial x_n}V^{SO}(\mathbf{x},t)}{a_n}\right)^2}{2\alpha_1} + \tag{52}$$

$$\sum_{n=1}^{N} \frac{\frac{\partial}{\partial x_n}V^{SO}(\mathbf{x},t)}{a_n}\left[r - \frac{1}{\alpha_1}\left[\left[\alpha_2 - \omega_1 - \omega_2\left[\overline{x} - x_n\right] - \frac{\frac{\partial}{\partial x_n}V^{SO}(\mathbf{x},t)}{a_n}\right] + \sum_{m=1}^{N} c_{mn}[x_m - x_n]\right]\right];$$

$$V^{SO}(\mathbf{x},T) = 0. \tag{53}$$

By inspection, it is possible to see that a value function of the form

$$V^{SO}(\mathbf{x},t) = K^{SO}(t) + \sum_{n=1}^{N} K_n^{SO}(t)x_n + \sum_{n=1}^{N}\sum_{m=1}^{N} K_{mn}^{SO}(t)x_m, \tag{54}$$

satisfying $K^{SO}(T) = K_n^{SO}(T) = K_{mn}^{SO}(T) = 0$, solves the Hamilton-Jacobi-Bellman equation (53). Finding precise expressions for the time-dependent functions $K^{SO}, K_n^{SO}, K_{mn}^{SO}$ is conceptually straightforward (it is the solution of a system of linear differential equations), though computationally cumbersome. In light of Eq. (51), optimal extraction strategies satisfy

$$q_n^{SO}(\mathbf{x},t) = \frac{1}{\alpha_1}\left[\alpha_2 - \omega_1 - \omega_2\left[\overline{x} - x_n\right] - \frac{K_{nn}^{SO}(t)x_n + \sum_{m=1}^{N} K_{mn}^{SO}(t)x_n + K_n^{SO}(t)}{a_n}\right], \quad n = 1, 2, ..., N \tag{55}$$

and thus are linear in the state. We suppose that model primitives are such the linear strategy $\mathbf{q}^{\mathbf{SO}}$ does not violate the non-negativity constraint $\mathbf{x} \geq \mathbf{0}$. ■

Proof of Theorem 2. The following argument is based on results in Athanassoglou (2010). In contrast to the analysis of Theorem 1, we consider the Hamilton-Jacobi-Bellman sufficient conditions for a Markov perfect equilibrium, which appear in Theorem 4.4 of Dockner et al. (2000).

For $q_n^{SO}(\mathbf{x},t)$ to be the argmax of the right hand side of Eq. (31) requires

$$\frac{\frac{\partial}{\partial x_n}V_n(\mathbf{x},t)}{a_n} = -\alpha_1 q_n^{SO}(\mathbf{x},t) + \alpha_2 - \mu\left[\omega_1 - \omega_2\left[\overline{x} - x_n\right]\right], \tag{56}$$

with $q_n^{SO}(\mathbf{x}(t),t)$ linear in $\mathbf{x}$. We conjecture that V_n satisfies

$$V_n(\mathbf{x},t) = K_n(t) + \sum_{m=1}^{N} K_{mn}(t)x_m + \sum_{\ell=1}^{N}\sum_{m=1}^{N} K_{\ell mn}(t)x_\ell x_m, \tag{57}$$

for some time-dependent functions $K_{\ell mn}, K_{mn}$ that are consistent with the optimality requirement given by Eq. (56). Substituting Eq. (57) into the Hamilton-Jacobi-Bellman equations (31) yields

$$\begin{aligned}
&\sum_{\ell=1}^{N}\sum_{m=1}^{N}\big[\delta K_{\ell mn}(t) - \frac{\mathrm{d}}{\mathrm{d}t}K_{\ell mn}(t)\big]x_\ell x_m + \sum_{m=1}^{N}\big[\delta K_{mn}(t) - \frac{\mathrm{d}}{\mathrm{d}t}K_{mn}(t)\big]x_m + \delta K_n(t) - \frac{\mathrm{d}}{\mathrm{d}t}K_n(t)\\
&= \frac{-\alpha_1 q_n^{SO}(\mathbf{x},t)^2}{2} + (\alpha_2 - \mu\omega_1)q_n^{SO}(\mathbf{x},t) - \mu\omega_2 q_n^{SO}(\mathbf{x},t)(\overline{x} - x_n) + \phi_n(\mathbf{x},t)\\
&\quad + \Big[-\alpha_1 q_n^{SO}(\mathbf{x},t) + \alpha_2 - \mu\omega_1 - \mu\omega_2\big[\overline{x} - x_n\big]\Big]\Big[r - q_n^{SO}(\mathbf{x},t) + \sum_{m=1}^{N} c_{mn}[x_m - x_n]\Big]\Big]\\
&\quad + \sum_{m\neq n} \tfrac{K_{mmn}(t)x_m + \sum_{\ell=1}^{N}K_{\ell mn}(t)x_\ell + K_{mn}(t)]}{a_m}\Big[r - q_m^{SO}(\mathbf{x},t) + \sum_{\ell=1}^{N} c_{\ell m}[x_\ell - x_m]\Big]\Big], \ t < T;
\end{aligned} \tag{58}$$

$$\sum_{\ell=1}^{N}\sum_{m=1}^{N} K_{\ell mn}(T)x_\ell x_m + \sum_{m=1}^{N} K_{mn}(T)x_m + K^n(T) = \phi_n(\mathbf{x},T), \quad t = T. \tag{59}$$

To simplify Hamilton-Jacobi-Bellman conditions (58), let $g_n(\mathbf{x},t)$ and $g_n^T(\mathbf{x})$ denote the following functions

$$\begin{aligned}
g_n(\mathbf{x},t) \;&=\; \sum_{\ell=1}^{N}\sum_{m=1}^{N}\big[\delta K_{\ell mn}(t) - \frac{\mathrm{d}}{\mathrm{d}t}K_{\ell mn}(t)\big]x_\ell x_m + \sum_{m=1}^{N}\big[\delta K_{mn}(t) - \frac{\mathrm{d}}{\mathrm{d}t}K_{mn}(t)\big]x_m - \\
&\Big\{\frac{-\alpha_1 q_n^{SO}(\mathbf{x},t)^2}{2} + [\alpha_2 - \mu\omega_1]q_n^{SO}(\mathbf{x},t) - \mu\omega_2 q_n^{SO}(\mathbf{x},t)[x^0 - x_n] + \\
&\Big[-\alpha_1 q_n^{SO}(\mathbf{x},t) + \alpha_2 - \mu\omega_1 - \mu\omega_2\big[\overline{x} - x_n\big]\Big]\Big[r - q_n^{SO}(\mathbf{x},t) + \sum_{m=1}^{N} c_{mn}[x_m - x_n]\Big]\Big] + \\
&\sum_{m\neq n} \tfrac{K_{mmn}(t)x_m + \sum_{\ell=1}^{N}K_{\ell mn}(t)x_\ell + K_{mn}(t)}{a_m}\Big[r - q_n^{SO}(\mathbf{x},t) + \sum_{\ell=1}^{N} c_{\ell m}[x_\ell - x_m]\Big]\Big]\Big\}, t < T;
\end{aligned} \tag{60}$$

$$g_n^T(\mathbf{x}) \;=\; \sum_{\ell=1}^{N}\sum_{m=1}^{N} K_{\ell mn}(T)x_\ell x_m + \sum_{m=1}^{N} K_{mn}(T)x_m. \tag{61}$$

Using Eqs. (60) and (61), the Hamilton-Jacobi-Bellman conditions given by (58) and (59) may be rewritten as

$$
\begin{aligned}
g_n(\mathbf{x},t) + \delta K_n(t) - \frac{\mathrm{d}}{\mathrm{d}t}K_n(t) &= \phi_n(\mathbf{x},t), \quad t<T \\
g_n^T(\mathbf{x}) + K_n(T) &= \phi_n(\mathbf{x},T).
\end{aligned} \tag{62}
$$

Given Eq. (51) and the linear structure of the optimal extraction strategies, the functions $g_n(\mathbf{x},t)$ are second-degree polynomials in $\mathbf{x}$. Thus, they equal their Taylor expansions, with respect to $\mathbf{x}$, about any point. Consider then the polynomial mechanism $\boldsymbol{\phi}^{P,SO} = (\phi_1^{P,SO}, \phi_2^{P,SO}, ..., \phi_N^{P,SO})'$ given by the non-constant part of the Taylor expansion of $g_n(\mathbf{x},t)$ and $g_n^T(\mathbf{x})$, about points $(\mathbf{x^{SO}}(t),t)$ and $(\mathbf{x^{SO}}(T),T)$ respectively (recall that $\mathbf{x^{SO}}(t)$ denotes the water-table path corresponding to $\mathbf{q^{SO}}$). Letting $\nabla_x f(\mathbf{x},\mathbf{y})$ and $\nabla^2_{xx} f(\mathbf{x},\mathbf{y})$ respectively denote the vector of partial derivatives and matrix of second partial derivatives of f with respect to $\mathbf{x}$ evaluated at $(\mathbf{x},\mathbf{y})$, we obtain:

$$
\phi_n^{P,SO}(\mathbf{x},t) = \begin{cases}
\left[\mathbf{x} - \mathbf{x^{SO}}(t)\right]' \nabla_x g_n(\mathbf{x^{SO}}(t),t) & \\
+\frac{1}{2}[\mathbf{x} - \mathbf{x^{SO}}(t)]' \nabla^2_{xx} g_n(\mathbf{x^{SO}}(t),t)[\mathbf{x} - \mathbf{x^{SO}}(t)] & t<T \\
\left[\mathbf{x} - \mathbf{x^{SO}}(T)\right]' \nabla_x g_n^T(\mathbf{x^{SO}}(T)) & \\
+\frac{1}{2}[\mathbf{x} - \mathbf{x^{SO}}(T)]' \nabla^2_{xx} g_n^T(\mathbf{x^{SO}}(T))[\mathbf{x} - \mathbf{x^{SO}}(T)] & t=T.
\end{cases} \tag{63}
$$

The specification of mechanism (63) in combination with Taylor's theorem implies that

$$
g_n(\mathbf{x},t) = \phi_n^{P,SO}(\mathbf{x},t) + g_n(\mathbf{x^{SO}}(t),t), \quad \text{for all } (\mathbf{x},t), \quad t<T; \tag{64}
$$

$$
g_n^T(\mathbf{x}) = \phi_n^{P,SO}(\mathbf{x},T) + g_n^T(\mathbf{x^{SO}}(T)), \quad \text{for all } \mathbf{x}. \tag{65}
$$

Thus, applying the mechanism (63) reduces Eq. (62) to the following ordinary differential equation and terminal condition

$$
\delta K_n(t) = \frac{\mathrm{d}}{\mathrm{d}t}K_n(t) - g_n(\mathbf{x^{SO}}(t),t), \quad t<T; \tag{66}
$$

$$
K_n(T) = -g_n^T(\mathbf{x^{SO}}(T)), \tag{67}
$$

which has a unique solution $K_n(t)$. Substituting this into the value function given by Eq. (58), repeating this argument for users $m \neq n$, and invoking Theorem 4.4 in Dockner et al. (2000) establishes that $\mathbf{q^{SO}}$ is induced in Markov perfect equilibrium.

A second-degree polynomial mechanism that induces the socially optimal extraction strategy $\mathbf{q}^{SO}$ in Markov perfect equilibrium may not be unique since functions K_{lmn} and K_{mn} of Eq. (57) are not uniquely defined. Instead, they are only required to be consistent with Eq. (56) for every user. Different (appropriate) choices of K_{lmn} and K_{mn} may lead to different specifications of g_n and K_n and, consequently, different equilibrium mechanisms for inducing $\mathbf{q}^{SO}$. ∎

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