A simple mechanism for a complex aquifer

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Abstract

Standard economic models of groundwater management assume perfect transmissivity (i.e., the aquifer behaves as a bathtub), no external effects of groundwater stocks, and/or homogenous agents. In this article, we develop a model relaxing these assumptions. Although our model generalizes to an arbitrary number of cells, we are able to obtain key insights with a two-cell finite-horizon differential game. We find a simple linear mechanism that induces the socially optimal extraction path in Markov-perfect equilibrium. Moreover, implementation requires that the regulator need only monitor the state of the resource (groundwater elevation), not individual extraction rates. We illustrate the mechanism with a simulation based on data from the Indian state of Andhra Pradesh. The simulation suggests that notable welfare loss may occur if the regulator disregards physical and economic complexity.

Keywords: common property resource, differential games, groundwater extraction, imperfect monitoring, Markov perfect equilibrium

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Introduction

Over the past four decades, a large economic literature has developed on optimal aquifer management. An important assumption used in much of this work is that an aquifer behaves like a one-dimensional "bathtub." In a bathtub model, water flows to the lowest point instantaneously and the water table is level throughout. Despite this assumption's mathematical convenience, aquifers are not underground caves filled with water, but rather saturated materials such as porous rock. As a result, transmissivity (horizontal flow) is lower than in a bathtub and groundwater depth can vary across space.[1]

In a bathtub model, spatial considerations are unimportant. With limited transmissivity, however, location matters. Cones of depression develop around individual wells, and the impact of extraction on other users decreases in distance from individual wells. Water-extracting agents are not uniformly located on the land overlying the water reserves. Instead, they tend to be found in discrete clusters. Thus, a modeling assumption ignoring these effects can be expected to yield results of questionable validity.

Literature abandoning the bathtub model in favor of more realistic dynamics has avoided strategic interaction among agents (Brozović et al., 2006; Chakravorty and Umetsu, 2003; Zeitouni and Dinar, 1997), and/or has assumed identical agents (Khalatbari, 1977; Eswaran and Lewis, 1984).[2] In addition, while these studies compare unregulated equilibrium outcomes with socially optimal outcomes, they do not derive pricing mechanisms that attain the social optimum.

The importance of accounting for spatial complexity and heterogeneous agents increases further if equity is considered. In developing countries, agriculture is often characterized by large land holdings of relatively wealthy owners and small tracts worked by poorer households. Distributional welfare analysis in such settings requires a model of water extraction that approximates actual water table dynamics while allowing for strategic behavior among heterogeneous agents.

Our work incorporates another element overlooked in the previous literature: stock externalities. Groundwater extraction does not necessarily take place in environmental or political-economic isolation. The level of the groundwater stock may have costs felt beyond the users themselves. Environmental impacts may include effects on nearby wetlands, land subsidance, or saltwater intrusion in coastal areas. Political effects (our present focus) may be felt if farmers' variable pumping costs are borne by the state. For example, it is common in many developing countries for agricultural electricity, the primary variable input to extraction, to be provided either for free or with a lump-sum tariff. India represents a salient example of the above policies and their complex (and at times controversial) economic and environmental effects (see Dubash, 2007; Shah, 2008; World Bank, 2001). In general, the lower the water table, the more energy is required to extract the water. A stock externality is thus generated to the extent that electricity costs are borne by the state rather than users.

We advance the literature by developing a spatially complex model of groundwater extraction addressing the above considerations (our results are still subject to such common simplifying assumptions as perfectly rational agents and commonly known deterministic hydrology). Within this setting, we derive competitive equilibrium and socially-optimal extraction paths and investigate a simple class of policy instruments. Specifically, since continuous monitoring of agent extraction rates is likely to be infeasible, we only allow governmental transfers to be made on the stock of the resource. In this sense, the regulatory problem is similar to that of non-point source pollution (e.g., Segerson, 1988; Xepapadeas, 1992; Karp, 2005).

As exceedingly complex instruments are unlikely to be appealing in practice, we restrict attention to simple linear additive mechanisms (a linear tax based on water table depth). Interestingly, imposing such severe restrictions on policy instruments has no adverse effect on social welfare. In contrast to the steady state analysis of Xepapadeas (1992) and Karp (2005), we exhibit a simple price mechanism that exactly induces the socially optimal extraction *path* in Markov-perfect equilibrium. This positive result holds regardless of the number and spatial configuration of extracting agents as well as the spatial and physical characteristics of the aquifer (number of cells, connectivity).

To illustrate the importance of these modeling techniques we conduct comparative simulations based on aquifer characteristics in the Indian state of Andhra Pradesh. In a simple two player game we derive the socially optimal extraction path and exhibit the linear transfer schemes that induce it in equilibrium. We further investigate the policy implications of improperly using a bathtub model and of failing to allow for agent heterogeneity. Our findings suggest that substantial welfare loss may result from implementing policy that is predicated on incorrect physical and technological assumptions.

The paper is organized as follows. Focusing initially on the two-cell case, in the next two sections we introduce our model and analyze the socially optimal solution. In the following section, we discuss simple policy tools and prove that they can induce any feasible and continuously differentiable extraction path (including the social optimum) in Markov-perfect equilibrium with zero transfers. We then conduct a numerical simulation that applies our theoretical insights to a simple groundwater game set in rural India. We conclude with a discussion of results and useful extensions. The Appendix generalizes the model to an arbitrary number of cells and agents.

Model

We model the aquifer as a common property resource not characterized by open access (the number and location of agents is fixed over time). Consider two adjacent aquifer cells, indexed by $i = 1, 2$, each having a single agent.[3]

Let $\overline{x}_i$ denote the initial water table elevation for cell i, $x_i(t) \in \Re_+$ denote elevation at time t. The extraction rate at time t for agent i is $q_i(t)$. Departing from the one-dimensional bathtub model, the water tables of agents 1 and 2 follow the set of differential equations (a dot indicates a derivative with respect to time):

$$a_i \cdot \dot{x}_i = r - q_i + c[x_j - x_i], \quad i = 1, 2 \ \ i \neq j.^4 \tag{1}$$

Variations of these dynamics appear in Eswaran and Lewis (1984); Khalatbari (1977); Zeitouni and Dinar (1997). Here, r is the (uniform) rate of recharge, and a_i is the surface area of agent i's land multiplied by the storativity coefficient of agent i's micro-watershed (which depends on unspecified geological factors).[5] Parameter c is the transmissivity between adjacent aquifer cells, a measure of the "connectivity" between the agents. The term $c[x_j - x_i]$ is the water flux between agent i and j's micro-watersheds.[6]

Each agent's profit function, $h_i(q_i(t))$, is twice continuously differentiable and strictly concave. Profit depends only on extraction; all stock effects are external to the agents.

Solving the system of differential equations (1) yields water table levels at time t, as functions of initial conditions, $\overline{x}_i$, and the extraction history $q_i(s)$ for $0 \leq s \leq t$. Letting $c_i \equiv c/a_i$, agent i's water table level at time t is:

$$\begin{aligned} x_i(t) \;=\; & \frac{1}{a_1 + a_2} \Big\{ \overline{x}_i \left[a_i + a_j \mathrm{e}^{-[c_1+c_2]t} \right] + \overline{x}_j a_j \left[1 - \mathrm{e}^{-[c_1+c_2]t} \right] + \\ & \int_0^t \left[[r - q_i(s)] \left[1 + \frac{a_j}{a_i} \mathrm{e}^{[c_1+c_2][s-t]} \right] + [r - q_j(s)] \left[1 - \mathrm{e}^{[c_1+c_2][s-t]} \right] \right] \mathrm{d}s \Big\} . \end{aligned} \tag{2}$$

Lemma 1 shows how these dynamics nest the extreme cases of unconnected cells and a bathtub if the extraction rate is bounded and does not change too quickly.

Lemma 1 *Let $q_i(t;c)$ and $x_i(t;c)$ denote the extraction rate and water table at time t as functions of c. If extraction paths $q_i(t)$ are bounded and differentiable with bounded derivatives, water table dynamics defined by Eq. (1): (i) approach $\dot{x}_i = \frac{r - q_i(t;0)}{a_i}$ as $c \to 0$; and (ii) approach $\dot{x}_i = \frac{2r - q_1(t;\infty) - q_2(t;\infty)}{a_1 + a_2}$ as $c \to \infty$.*

Proof. (i) Letting $\lim_{c \to 0} q_i(t;0) = q_i(t;0)$, the dynamics of an unconnected aquifer are

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\lim_{c \to 0} x_i(t;c) \right] = \lim_{c \to 0} \frac{\mathrm{d}}{\mathrm{d}t} \left[x_i(t;c) \right] = \frac{r - q_i(t;0)}{a_i}, \text{ for all } t \in [0,T]. \tag{3}$$

Taking limits as $c \to 0$, Eqs. (1) and (2) arrive at this expression.

(ii) Letting $\lim_{c\to\infty} q_i(t;c) = q_i(t;\infty)$, the dynamics of a single-cell aquifer are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\lim_{c\to\infty} x_i(t;c)\right] = \lim_{c\to\infty}\frac{\mathrm{d}}{\mathrm{d}t}\left[x_i(t;c)\right] = \frac{2r - q_1(t;\infty) - q_2(t;\infty)}{a_1+a_2}, \text{ for all } t\in[0,T]. \qquad (4)$$

Consider the water table equations $x_i(t,c)$ given by Eq. (2). Since the function $q_i(t,c)$ is bounded, the Bounded Convergence Theorem (see Rudin, 1976) implies that the integral of the limit is equal to the limit of the integral. Taking the limit as $c\to\infty$ in Eq. (2) yields:

$$\lim_{c\to\infty} x_i(t;c) = \frac{\overline{x}_i + \overline{x}_j + 2rt - \lim_{c\to\infty}\int_0^t q_1(s;\infty) + q_2(s;\infty)\mathrm{d}s}{a_1+a_2}.$$

Differentiating with respect to t,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\lim_{c\to\infty} x_i(t;c)\right] = \frac{2r - q_1(t;\infty) - q_2(t;\infty)}{a_1+a_2}.$$

Substituting $x_i(t;c)$ and $x_j(t;c)$ from Eq. (2) into agent i's dynamics yields:

$$\begin{aligned}
a_i\dot{x}_i(t;c) &= r - q_i(t;c) + \tfrac{c}{a_1+a_2}\Bigg\{\mathrm{e}^{-[c_1+c_2]t}\left[\overline{x}_j a_j\left[1+\tfrac{a_i}{a_j}\right] - \overline{x}_i a_i\left[1+\tfrac{a_j}{a_i}\right]\right] \\
&\quad + \int_0^t \mathrm{e}^{[c_1+c_2][s-t]}\left[r\left[\tfrac{a_i}{a_j}-\tfrac{a_j}{a_i}\right] + q_i(s;c)\left[1+\tfrac{a_j}{a_i}\right] - q_j(s;c)\left[1+\tfrac{a_i}{a_j}\right]\right]\mathrm{d}s\Bigg\} \\
&= r\left[\tfrac{2a_i}{a_1+a_2} - \tfrac{a_i^2-a_j^2}{[a_1+a_2]^2}\mathrm{e}^{-[c_1+c_2]t}\right] \\
&\quad - q_i(t;c) + \tfrac{c}{a_1+a_2}\Bigg\{\mathrm{e}^{-[c_1+c_2]t}\left[\overline{x}_j a_j\left[1+\tfrac{a_i}{a_j}\right] - \overline{x}_i a_i\left[1+\tfrac{a_j}{a_i}\right]\right] \\
&\quad + \int_0^t \mathrm{e}^{[c_1+c_2][s-t]}\left[q_i(s;c)\left[1+\tfrac{a_j}{a_i}\right] - q_j(s;c)\left[1+\tfrac{a_i}{a_j}\right]\right]\mathrm{d}s\Bigg\} \\
&= r\left[\tfrac{2a_i}{a_1+a_2} - \tfrac{a_i^2-a_j^2}{[a_1+a_2]^2}\mathrm{e}^{-[c_1+c_2]t}\right] \\
&\quad - q_i(t;c) + \tfrac{c}{a_1+a_2}\mathrm{e}^{-[c_1+c_2]t}\left[\overline{x}_j a_j\left[1+\tfrac{a_i}{a_j}\right] - \overline{x}_i a_i\left[1+\tfrac{a_j}{a_i}\right]\right] \\
&\quad + \tfrac{a_j}{a_1+a_2}\left[q_i(t;c) - \mathrm{e}^{-[c_1+c_2]t}q_i(0;c) - \int_0^t \mathrm{e}^{[c_1+c_2][s-t]}\tfrac{\mathrm{d}}{\mathrm{d}s}[q_i(s;c)]\mathrm{d}s\right] \\
&\quad - \tfrac{a_i}{a_1+a_2}\left[q_j(t;c) - \mathrm{e}^{-[c_1+c_2]t}q_i(0;c) - \int_0^t \mathrm{e}^{[c_1+c_2][s-t]}\tfrac{\mathrm{d}}{\mathrm{d}s}[q_j(s;c)]\mathrm{d}s\right].
\end{aligned}$$

Taking the limit of this expression as $c\to\infty$ and applying the Bounded Convergence Theorem yields the desired result. ■

Social Optimum

A social planner wishes to maximize the net benefit of water extraction: the discounted (at rate $\delta > 0$) sum of agent profit less social damages (e.g., cost of energy used in extraction), denoted $D(\mathbf{q}(t), \mathbf{x}(t))$, where $\mathbf{q}(t) \equiv (q_1(t), q_2(t))'$ and $\mathbf{x}(t) \equiv (x_1(t), x_2(t))'$. This damage function is increasing in q_i, decreasing in x_i, and convex in all its arguments. The terminal time is T, and the "scrap value" of the aquifer is $-D^T(\mathbf{x}(T))$. We restrict attention to cases in which the social optimum does not involve exhaustion of the resource for any agent before time T. Initial conditions are $\overline{\mathbf{x}} \equiv (\overline{x}_1, \overline{x}_2)'$. The social planner's optimal control problem is

$$\begin{aligned} \max_{\mathbf{q}(t)} \quad & \int_0^T \mathrm{e}^{-\delta t} \left[\sum_{i=1}^{2} h_i(q_i(t)) - D(\mathbf{q}(t), \mathbf{x}(t)) \right] \mathrm{d}t - \mathrm{e}^{-\delta T} D^T(\mathbf{x}(T)) \\ \text{subject to:} \quad & a_i \dot{x}_i = r - q_i + c[x_j - x_i], \quad i, j = 1, 2; i \neq j \\ & \mathbf{x}(0) = \overline{\mathbf{x}}. \end{aligned} \tag{5}$$

Letting $\boldsymbol{\lambda}(t) \equiv (\lambda_1(t), \lambda_2(t))'$ denote costate variables, the current-value Hamiltonian is:

$$H(\mathbf{q}(t), \mathbf{x}(t), \boldsymbol{\lambda}(t), t) = \sum_{i=1}^{2} h_i(q_i(t)) - D(\mathbf{q}(t), \mathbf{x}(t)) + \sum_{\substack{i=1 \\ j \neq i}}^{2} \frac{\lambda_i(t)}{a_i} \left[r - q_i(t) + c[x_j(t) - x_i(t)] \right].$$

The necessary conditions for an interior solution is

$$\frac{\mathrm{d}h_i(q_i(t))}{\mathrm{d}q_i} - \frac{\partial D}{\partial q_i} - \frac{\lambda_i(t)}{a_i} = 0, \text{ for } i = 1, 2. \tag{6}$$

Optimal conditions for the co-state variables yield the following differential equations:

$$\dot{\lambda}_i(t) = [\delta + c_i] \lambda_i(t) - c_j \lambda_j(t) + \frac{\partial D}{\partial x_i} \text{ for } i, j = 1, 2; i \neq j. \tag{7}$$

Assuming an interior optimal solution in which the aquifer is never exhausted, transversality conditions $\lambda_i(T) = -\partial D^T(x_i(T))/\partial x_i$ imply

$$\frac{\mathrm{d}h_i(q_i(T))}{\mathrm{d}q_i} = \frac{\partial D(\mathbf{q}(T), \mathbf{x}(T))}{\partial q_i} + \frac{\partial D^T(\mathbf{x}(T))}{\partial x_i}, \quad \text{for } i = 1, 2, \tag{8}$$

providing terminal conditions for extraction rates and water table levels.

Conditions (6), (7), and (8) are necessary and sufficient for a strictly interior optimum in which the aquifer is not exhausted (see Sethi and Thompson, 2000). Differentiating Eq. (6) with respect to t yields,

$$\frac{\dot{\lambda}_i(t)}{a_i} = \left[\frac{\mathrm{d}^2 h_i}{\mathrm{d}q_i^2} - \frac{\partial^2 D}{\partial q_i^2}\right]\dot{q}_i(t) - \frac{\partial^2 D}{\partial q_i \partial q_j}\dot{q}_j(t) - \frac{\partial^2 D}{\partial q_i \partial x_i}\dot{x}_i(t) - \frac{\partial^2 D}{\partial q_i \partial x_j}\dot{x}_j(t). \tag{9}$$

Substituting Eqs. (6) and (9) into (7), and rewriting the stock dynamics given by Eq. (1), we obtain four differential equations involving $\mathbf{q}(t)$ and $\mathbf{x}(t)$. This system, together with initial conditions on the water stocks and terminal conditions on the extraction rates, specifies the socially optimal extraction and water stock paths $\langle \mathbf{q}^{SO}(t), \mathbf{x}^{SO}(t)\rangle$.

Policy Analysis

We suppose the regulator does not have resources to monitor agents' extraction decisions, but can costlessly monitor the state variables $\mathbf{x}(t)$. This scenario is analogous to a dynamic nonpoint source stock pollution problem in which the regulator can monitor ambient pollution levels but not individual emissions (e.g., Xepapadeas, 1992). The regulator is further restricted in that the only policy tools at her disposal are linear transfers, $\beta(t)$ for $t < T$ and β^T for $t = T$. In Theorem 1 below, we show that despite these restrictions the regulator can induce the socially optimal path in Markov perfect equilibrium with a mechanism $\boldsymbol{\phi}(\mathbf{x}(t), t) \equiv (\phi_1(x_1(t), t), \phi_2(x_2(t), t))'$, such that

$$\phi_i(x_i(t), t) = \begin{cases} \beta_i(t)[x_i(t) - x_i^{SO}(t)] & \text{for } t < T \\ \beta_i^T[x_i(t) - x_i^{SO}(t)] & \text{for } t = T. \end{cases} \tag{10}$$

Before proving this result, it is useful to discuss Markov equilibria for linear mechanisms.

A mechanism $\boldsymbol{\phi}(\mathbf{x}(t), t)$ induces a differential game between the agents. Given a strategy $q_j^*(\mathbf{x}(t), t)$ chosen by agent j, agent i chooses as his strategy the solution to:[7]

$$
\begin{aligned}
\max_{q_i(t)\geq 0} \quad & \int_0^T \mathrm{e}^{-\delta t}[h_i(q_i(t)) + \beta_i(t)[x_i - x_i^{SO}]\mathrm{d}t + \mathrm{e}^{-\delta T}\beta_i^T[x_i(T) - x_i^{SO}(T)] \\
\text{subject to:} \quad & a_i \dot{x}_i(t) = r - q_i(t) + c[x_j(t) - x_i(t)] \\
& a_j \dot{x}_j(t) = r - q_j^*(\mathbf{x}(t), t) + c[x_i(t) - x_j(t)] \\
& \mathbf{x}(0) = \overline{\mathbf{x}}.
\end{aligned}
\tag{11}
$$

An open-loop strategy is one in which agents pre-commit to an entire extraction path at the beginning of the game, and so is not a function of current state variables. Formally, a strategy $q_i(\mathbf{x}(t), t)$ is open-loop, if $q_i^*(\mathbf{x}(t), t) = q_i^*(t)$ for all $\mathbf{x}(t) \in \Re^2$. An open-loop Nash equilibrium (defined below) is relatively simple to compute for this game.

Definition 1 *A set $(q_1^*(t), q_2^*(t))$ of open-loop strategies where $q_i^*(t) : [0, T] \mapsto \Re$, is an open-loop Nash equilibrium if, for each $i \in \{1, 2\}$ an optimal control path $q_i(t)$ of the maximization problem given by (11) exists and is given by $q_i(t) = q_i^*(t)$.*

In general, open-loop Nash equilibria are restrictive since they do not allow agents to adapt strategies to changes in the state vector. This equilibrium concept is typically justifiable only if the state vector is unobservable over time, rendering moot the ability to adapt.

Markov-perfect equilibrium (defined below) is an alternative concept that overcomes this shortcoming. An agent choosing a Markov strategy conditions his current extraction only on the value of the current state variable (not otherwise on the game's previous history).[8]

Definition 2 *Let $\mathbf{x}(t) \in X \subseteq \Re^2$ for all $t \in [0, T]$. A set $(q_1^*(\mathbf{x}(t), t), q_2^*(\mathbf{x}(t), t))$ of Markovian strategies where $q_i^*(\mathbf{x}(t), t) : X \times [0, T] \mapsto \Re$, is a Markovian-Nash equilibrium if, for each $i \in \{1, 2\}$ an optimal control path $q_i(t)$ of the maximization problem given by (11) exists and is given by $q_i(t) = q_i^*(\mathbf{x}(t), t)$. A Markov-perfect equilibrium is a subgame-perfect Markovian-Nash equilibrium.*

Identifying a Markov-perfect equilibrium typically requires the solution of a complex system of Hamilton-Jacobi-Bellman equations. The game considered here, however, has a structure that simplifies calculation of Markov-perfect equilibria. Specifically, it is a linear state game (as defined by Dockner et al., 2000) since (a) its objective functionals and state dynamics are linear in the state and (b) there are no cross terms of the sort $q_i x_i$ involving control and state variables. Dockner et al. (2000) (pp. 187-89) show that all open-loop Nash equilibria of linear state games are Markov-perfect.

The following proposition characterizes a Markov-perfect equilibrium induced by the linear mechanism described above.

Proposition 1 *The differential game (11) induced by mechanism $\boldsymbol{\phi}(\mathbf{x}(t), t)$ has a unique Markov-perfect equilibrium in open loop strategies $\mathbf{q}^*(t)$. For an interior solution, it satisfies:*

$$\frac{\mathrm{d}h_i(q_i^*(t))}{\mathrm{d}q_i} - \frac{f_i^{\phi}(t)}{a_i} = 0; \text{ where} \tag{12}$$

$$f_i^{\phi}(t) = \int_t^T \beta_i(s) \mathrm{e}^{\delta[t-s]} \frac{a_i + a_j \mathrm{e}^{[c_1+c_2][t-s]}}{a_1 + a_2} \mathrm{d}s + \beta_i^T \mathrm{e}^{\delta[t-T]} \frac{a_i + a_j \mathrm{e}^{[c_1+c_2][t-T]}}{a_1 + a_2}. \tag{13}$$

Proof. Let $\boldsymbol{\lambda}_i(t) = (\lambda_i^1, \lambda_i^2)'$ denote the costate variables for agent i corresponding to the state variables for agents 1 and 2. For an open-loop Nash equilibrium, the current-value Hamiltonian of agent i is:

$$\begin{aligned} H_i(q_i(t), \mathbf{x}(t), \boldsymbol{\lambda}_i(t), t) &= h_i(q_i) + \beta_i(t)[x_i(t) - x_i^{SO}(t)] + \\ & \sum_{k=1, j\neq k}^{2} \frac{\lambda_i^k(t)\,[r - q_k(t) + c[x_j(t) - x_k(t)]]}{a_k}. \end{aligned} \tag{14}$$

The necessary conditions for an interior solution for $i = 1, 2$ and $j \neq i$ are:

$$\frac{\mathrm{d}h_i(q_i^*(t))}{\mathrm{d}q_i} = \frac{\lambda_i^i(t)}{a_i} \tag{15}$$

$$\dot{\lambda}_i^i(t) = [\delta + c_i]\lambda_i^i(t) - c_j \lambda_j^i(t) - \beta_i(t) \tag{16}$$

$$\dot{\lambda}_i^j(t) = [\delta + c_j]\lambda_i^j(t) - c_i \lambda_i^i(t), \tag{17}$$

with transversality conditions

$$\lambda_i^i(T) = \beta_i^T, \quad \lambda_i^j(T) = 0. \tag{18}$$

Since $H_i(\cdot)$ is jointly concave in $q_i(t)$ and $\mathbf{x}(t)$, these conditions are sufficient. Eqs. (16) and (17) are a linear system of ordinary differential equations. Imposing the transversality condition yields the unique solution for $i = 1, 2$ and $j \neq i$:

$$\lambda_i^i(t) = f_i^{\boldsymbol{\phi}}(t), \tag{19}$$

$$\lambda_i^j(t) = \int_t^T \frac{\beta_i(s)\mathrm{e}^{\delta[t-s]}\left[1 - \mathrm{e}^{[c_1+c_2][t-s]}\right] a_j}{a_1 + a_2} \mathrm{d}s + \frac{\beta_i^T \mathrm{e}^{\delta[t-T]}\left[1 - \mathrm{e}^{[c_1+c_2][t-T]}\right] a_j}{a_1 + a_2}. \tag{20}$$

Substituting $\lambda_i^i(t)$ into Eq. (15) obtains the desired result. Uniqueness follows from the assumption that h_i is strictly concave. ■

To interpret Proposition 1, it is useful to calculate the shadow value of a unit of water table height for agent i at time t given a mechanism $\boldsymbol{\phi}(\mathbf{x}(t), t)$. Eq. (2) implies for $s \in (t, T)$,

$$x_i(s) = \frac{1}{a_1 + a_2}\Big\{x_i(t)\left[a_i + a_j \mathrm{e}^{[c_1+c_2][t-s]}\right] + x_j(t) a_j \left[1 - \mathrm{e}^{[c_1+c_2][t-s]}\right] + \tag{21}$$

$$\int_t^s \left[[r - q_i(z)]\left[1 + \frac{a_j}{a_i}\mathrm{e}^{[c_1+c_2][z-s]}\right] + [r - q_j(z)]\left[1 - \mathrm{e}^{[c_1+c_2][z-s]}\right]\right] \mathrm{d}z\Big\}, \tag{22}$$

so

$$\frac{\partial}{\partial x_i(t)}\left[\int_t^T x_i(s)\mathrm{d}s\right] = \int_t^T \left[\frac{a_i + a_j \mathrm{e}^{[c_1+c_2][t-s]}}{a_1 + a_2}\right] \mathrm{d}s. \tag{23}$$

For $\boldsymbol{\phi}(\mathbf{x}(t), t)$, the price for each unit of $x_i(s)$ at time s is $\beta_i(s)$, and the price of x_i^T at time T is β_i^T. The shadow price, or present discounted value (at time t) of the stream of losses incurred from a marginal drop in the water table at time t, is then $f_i^{\boldsymbol{\phi}}(t)$ in Eq. (13). To convert the shadow value from a marginal change in water table height, x, to a marginal change in volume, q, it is necessary to divide by a. Thus, Eq. (12) states that in equilibrium agents set marginal profit from extraction equal to its shadow value.

For an isolated aquifer ($c = 0$), the term $[a_i + a_j]\mathrm{e}^{[c_1+c_2][t-s]}/[a_1 + a_2]$ reduces to unity, i.e., the full impact of extraction is on x_i. For the bathtub case ($c \to \infty$), it reduces to $a_i/[a_1+a_2]$: The impact is proportional to the agent's relative share of the aquifer.

We now show that mechanisms ϕ given by Eq. (10) can induce every feasible and continuously differentiable extraction path over $[0,T]$.

Theorem 1 *Let* $\{\hat{q}_i(t) : t \in [0,T]\}$, *be an arbitrary continuously differentiable feasible extraction path satisfying* $\frac{\mathrm{d}h_i(\hat{q}_i(t))}{\mathrm{d}q_i} < \infty$, *and* $q_i^{\boldsymbol{\phi}}(t)$ *be the unique open-loop Markov-perfect equilibrium extraction path induced by linear-state mechanism* $\boldsymbol{\phi}$. *If* $q_i^{\boldsymbol{\phi}}(t)$ *is everywhere interior, then there exists a unique mechanism such that* $q_i^{\boldsymbol{\phi}}(t) = \hat{q}_i(t)$ *for all* $t \in [0,T]$ *and* $i = 1,2$.

Proof. Since $h_i(\cdot)$ is strictly concave, it is sufficient to show that for any $\hat{q}_i(t)$, terms $\beta_i(t)$ and β_i^T of mechanism ϕ_i can be chosen such that

$$\frac{\mathrm{d}h_i(\hat{q}_i(t))}{\mathrm{d}q_i} = \frac{f_i^{\boldsymbol{\phi}}}{a_i}, \quad \text{for all } t \in [0,T]. \tag{24}$$

Suppose $\beta_i^T = a_i \mathrm{d}h_i(\hat{q}_i(T))/\mathrm{d}q_i$, ensuring that Eq. (24) is satisfied for $t = T$. Eq. (24) becomes

$$\int_t^T \beta_i(s)\mathrm{e}^{\delta[t-s]}\frac{a_i + a_j\mathrm{e}^{[c_1+c_2][t-s]}}{a_1+a_2}\mathrm{d}s \;=\; a_i\frac{\mathrm{d}h_i(\hat{q}_i(t))}{\mathrm{d}q_i} - \beta_i^T\mathrm{e}^{\delta[t-T]}\frac{a_i + a_j\mathrm{e}^{[c_1+c_2][t-T]}}{a_1+a_2}. \tag{25}$$

Performing the change of variable $z = T - t$, we have

$$-\int_0^z \beta_i(T-s)\mathrm{e}^{\delta[s-z]}\frac{a_i + a_j\mathrm{e}^{[c_1+c_2][s-z]}}{a_1+a_2}\mathrm{d}s = a_i\frac{\mathrm{d}h_i(\hat{q}_i(T-z))}{\mathrm{d}q_i} - \beta_i^T\mathrm{e}^{-\delta z}\frac{a_i + a_j\mathrm{e}^{-[c_1+c_2]z}}{a_1+a_2}. \tag{26}$$

Eq. (26) is a linear Volterra equation of the first kind with a kernel containing exponential functions, a general solution for which can be found in Polyanin and Manzhirov (2008)(p. 17). In our case, letting

$$g(z) = -a_i\frac{\mathrm{d}h_i(\hat{q}_i(T-z))}{\mathrm{d}q_i} + \beta_i^T\mathrm{e}^{-\delta z}\frac{a_i + a_j\mathrm{e}^{-[c_1+c_2]z}}{a_1+a_2},$$

the solution to (26) is given by

$$\beta_i(T-z) \;=\; \mathrm{e}^{-\delta z}\frac{\mathrm{d}}{\mathrm{d}z}\left\{\mathrm{e}^{-\frac{a_i[c_1+c_2]z}{a_1+a_2}}\int_0^z \frac{\mathrm{d}}{\mathrm{d}s}\left[g(s)\mathrm{e}^{[\delta+c_1+c_2]s}\right]\mathrm{e}^{\frac{a_j[c_1+c_2]s}{a_1+a_2}}\mathrm{d}s\right\}. \tag{27}$$

The assumed differentiability of $h_i(\cdot)$ and $\hat{q}_i(\cdot)$ ensures that Eq. (27) is well-defined. Repeating this argument for agent $j \neq i$ and collecting the $\beta_i(\cdot), \beta_j(\cdot)$ functions and β_i^T, β_j^T constants establishes the desired result. Uniqueness under an interior path follows from the fact that conditions (24) for $i \in \{1, 2\}$ are, in this case, necessary for the two paths to coincide. ■

We conclude this section with the following corollary to Theorem 1.

Corollary 1 *If* $\mathbf{q}^{SO}(t)$ *is continuously differentiable there exists a linear mechanism that induces it in Markov-perfect equilibrium with zero net transfers.*

The corollary follows directly from Theorem 1. If the socially optimal extraction path is continuously differentiable, and the regulator induces it with a mechanism $\boldsymbol{\phi}^{SO}$, then $\mathbf{x}(t) = \mathbf{x}^{SO}(t)$ for all t, and, by Eq. (10), $\boldsymbol{\phi}^{SO}(\mathbf{x}^{SO}(t), t) = \mathbf{0}$ for all t. Moreover, since $\mathbf{x}^{SO}(t) \geq \mathbf{0}$ for all t, the induced Markov-perfect equilibrium strategies satisfy the state non-negativity constraint.

Numerical Simulations

In this section, we numerically simulate the differential game (11) for two agricultural agents in a typical rural setting in semi–arid tropical India. In these regions, agricultural production was traditionally constrained by precipitation variations during the wet monsoon season. The advent of inexpensive pump technology in the 1970s coupled with subsidized electricity now allows year-round production (Shah, 2008; Reddy, 2005).

Table 1 lists the parameters used in the simulation.[9] We calculate monetary units in 2005 U.S. dollars, using the average annual exchange rate of 44 Rupees per dollar. Farmers are adjacent landholders with one hectare plots. They share a watershed that receives no recharge through lateral subsurface inflows over the boundary. As in the theoretical model, a hydraulic connection between the adjacent landholdings allows water to flow across this interface depending on the individual water table elevations. We assume homogeneous and isotropic aquifer properties and choose parameter values representative of subsurface properties of weathered crystalline rock found in large parts of peninsular India. We suppose constant characteristic values for the hydraulic transmissivity c. For both farmers, initial drawdown levels are at 20 meters below ground surface.

The agro-economic parameters are representative of small landholders growing paddy rice in two seasons per year. Each farmer pumps water from one borehole located on his plot. We specify restricted profit (net returns to land) as a quadratic function of water input for agents $i = 1, 2$:[10]

$$h_i(q_i) = \theta_i \left[\alpha_1 q_i + \alpha_2 q_i^2\right]. \tag{28}$$

Panel a of Figure 1 illustrates these profit functions for both farmers. Farmer 1 is more technically efficient in the sense that he can attain any feasible profit using less water than farmer 2.

Social costs reflect typical expenses for the state related to provision of rural energy and are presumed to be the same for both farmers. We assume a standard energy cost function,

$$D(\mathbf{q}, \mathbf{x}) = \sum_{i=1}^{2} q_i[d_1 + d_2[\bar{x} - x_i]], \tag{29}$$

where $\bar{x}$ denotes the elevation of the irrigated plot. Here, $\bar{x} - x_i$ is the the total drawdown for each agent at any given moment in time against which a certain quantity of water has to be lifted to the surface. We set the terminal time cost to $D^T(\mathbf{x}(T)) = 0$.

For all computations, we use Matlab with Simulink. We solve the system of differential equations (5) as a nonlinear programming problem using the control vector parameterization concept described in Becerra (2004) and the references therein. We utilize the Dormand-Prince formula fixed-step integration technique solver to obtain the socially optimal solution, with a seasonal discrete time-step over 10 years.

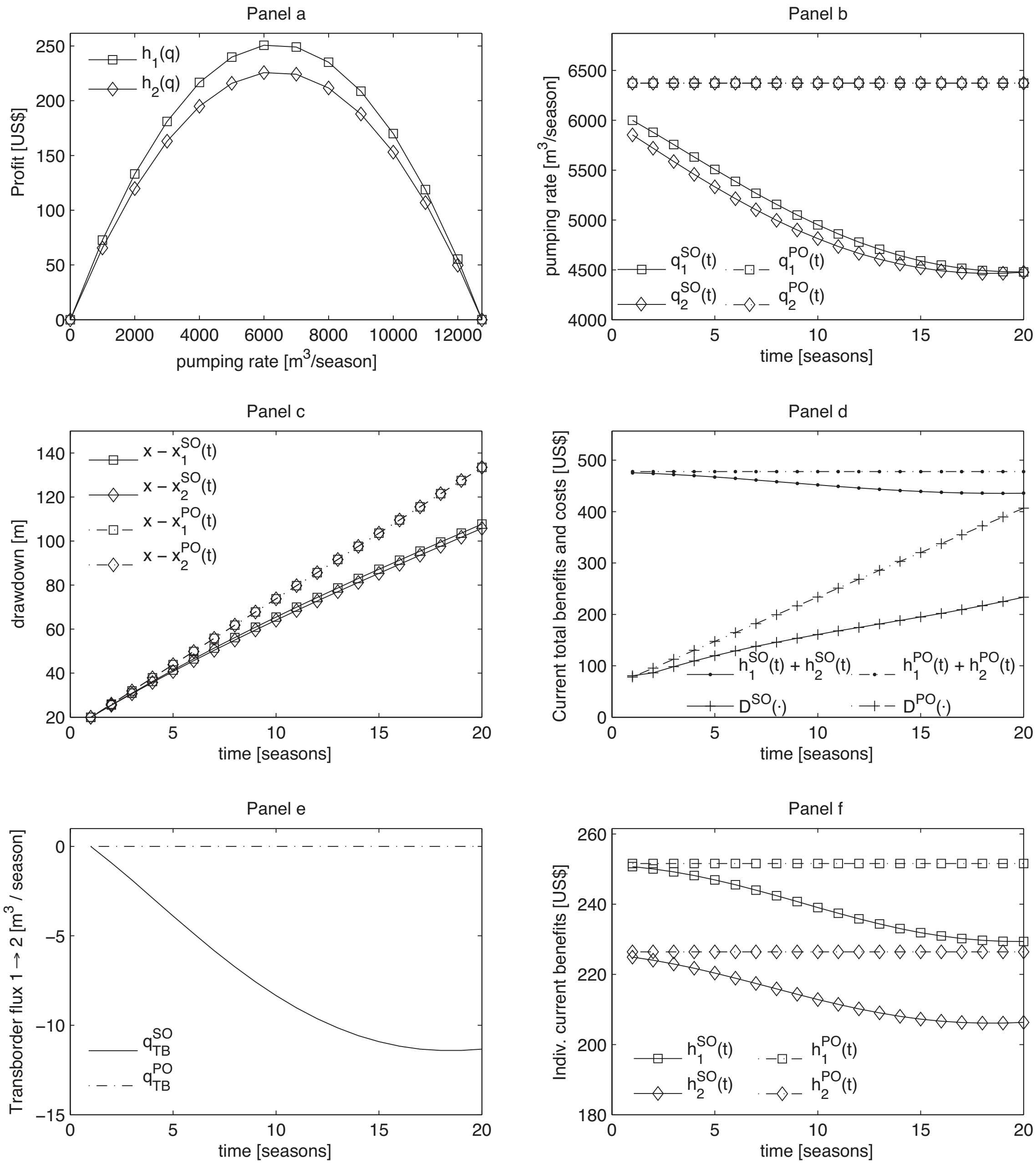

Figure 1: Simulation results for socially optimal (SO) and privately optimal (PO) extraction rates.

Parameter	Description	Value	Unit
A	landholding size	1	ha
S	effective porosity	0.01	n.a.
c	seasonal transmissivity	6	m^2
$\bar{x} - x_i(t)$	drawdown	20	m
r	seasonal recharge	0.04	m
δ	discount rate	0.05	n.a.
θ_1	productivity scaling factor	1	n.a.
θ_2	productivity scaling factor	0.9	n.a.
α_1	technological parameter	3.47	US$ / m^3
α_2	technological parameter	$-2.73 * 10^{-4}$	US$ / m^6
d_1	energy cost parameter	0.1	US$ / m^3
d_2	energy cost parameter	0.01	US$ / m^4

Table 1: Simulation parameter values

Socially optimal mechanism

Simulation results are shown in Figures 1 and 2. Socially optimal (SO) pumping rates decline over time (Panel b in Figure 1). Privately optimal (PO) pumping rates, representing the outcome of the unregulated status quo, are constant throughout the optimization period since extraction costs are not internalized by the agents.[11] Panel c illustrates how water tables decline at a slower rate at the social optimum, thus resulting in lower social damages (Panel d). Panel e shows the transborder flux in the micro-watershed between the two adjacent landholdings for both runs. A positive flux indicates a net subsurface water exchange from agent 1 to agent 2 and vice versa. We observe that throughout the simulation socially optimal extraction is such that water flows from agent 2 to agent 1.

Panel a in Figure 2 shows the development of the mechanism charges $\beta_1(t)$ and $\beta_2(t)$ to farmers 1 and 2, respectively, as a function of time with $t < T$. For the terminal time charges, we have $\beta_1^T = 1032.3$ and $\beta_2^T = 930.8$. Panel a suggests a surprising result: Dynamic mechanism charges initially take on negative values, implying that agents are, in theory, rewarded when water table levels are below socially optimal levels.

This somewhat counter-intuitive result has been observed in the context of optimal taxation of a stock pollutant (Benchekroun and van Long, 1998). It can be explained as follows.

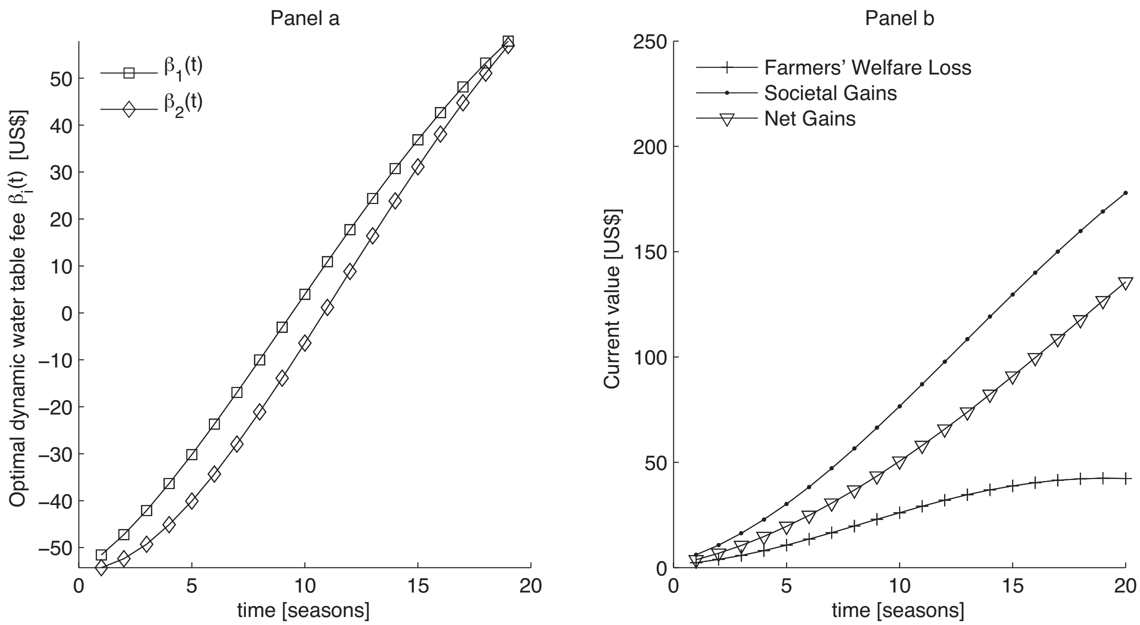

Figure 2: Simulation results: Optimal mechanism and its welfare effects

Suppose the dynamic charges were zero and the optimal extraction rate were constant over time. A large terminal charge can give agents an incentive to extract below the socially optimal rate for some $t < T$ if the marginal benefit of extraction at the social optimum were less than the discounted value of the terminal charge. A dynamic subsidy would thus be necessary to induce optimal extraction, and this subsidy would be increasing over time due to the effect of discounting. If the optimal extraction path were in fact decreasing over time, then the increase in the subsidy would be smaller than in the case of a constant extraction path. In our case, the socially optimal extraction path is decreasing quickly enough that it overcomes the effect of discounting such that the dynamic subsidy is actually decreasing over time, eventually becoming a tax.

The crossed curve in Panel b of Figure 2 shows total profit losses to farmers (in comparison to the unregulated status quo) due to their pumping at the socially optimal levels. These losses correspond to the difference between the dashed and solid dotted lines in Panel d of Figure 1. The dotted line in Panel b of Figure 2 shows the social cost savings achieved from reverting to socially optimal extraction. These savings correspond to the difference between the dashed and dotted crossed lines in Panel d in Figure 1. Throughout the simulation period, social cost savings are higher than farmer profit losses, with their net difference depicted by the upside-down-triangle line in Panel b of Figure 2. Hence, part of the overall social benefits can be redistributed to farmers to compensate their welfare losses. Such compensation may be important from the perspective of a real-world implementation since without it, farmers would not acquiesce to public policy of this sort unless they were coerced to do so.

Role of spatial and economic complexity

In this section we discuss welfare loss from making two kinds of mistakes in implementing the optimal mechanism: (i) incorrectly assuming that the underlying aquifer is a bathtub, or; (ii) incorrectly assuming that agents are homogeneous.

Under assumption (i), the regulator solves for the socially optimal extraction path assuming incorrectly that the aquifer has infinite transmissivity. In particular, she solves the optimal control problem given by Expression (5) with state dynamics given by Lemma 1. Then, she plugs the derived extraction path into Eq. (27), assuming that $c_1 = c_2 = \infty$, to obtain the mechanism charges. Implementation of this mechanism results in an induced equilibrium described by Proposition 1.

Figure 3 shows that aquifer dynamics have a major effect on optimal policy when transmissivity is low, a feature commonly found in hard rock or well consolidated sedimentary formations. The graph depicts percentage welfare loss (with regard to the social optimum) as a function of actual field transmissivity values c. The range over which c is varied ($0.3 - 2 * 10^4$ m^2/season) corresponds to field situations as reported in Raj (2004). The negative impact increases the less the aquifer resembles a bathtub in reality. As transmissivity increases, the bathtub assumption results in less welfare loss and eventually becomes innocuous.

Turning to assumption (ii), economic heterogeneity is defined as the ratio between the value of the two profit functions in Eq. (28). In our calculations, we take h_1 as given and vary θ_2 from 0.1 to 1. The regulator's mistake is now the following. First, she solves for

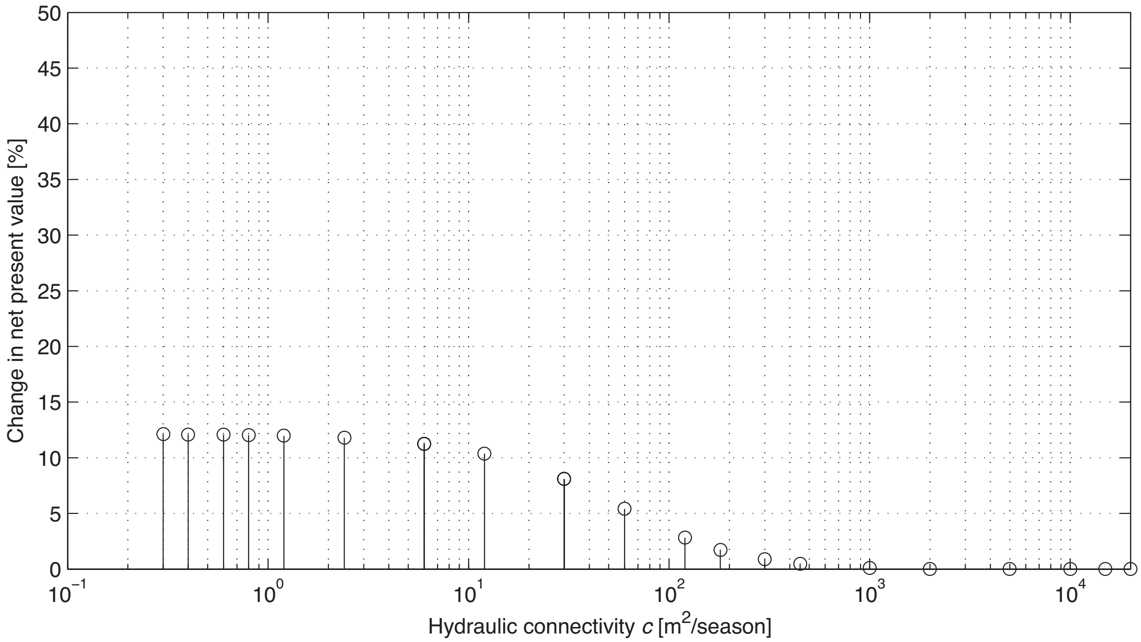

Figure 3: Percent social welfare loss from incorrect "bathtub" assumption.

the socially optimal extraction path assuming incorrectly that the two agents have identical profit functions. That is, she solves the optimal control problem given by Expression (5) supposing that $\theta_2 = 1$ in the objective function. Then, she plugs the derived extraction path into Eq. (27) to obtain the mechanism charges. The subsequent implementation of this mechanism results is an induced equilibrium, described by Proposition 1, that is suboptimal in relation to the social optimum, which is the solution to optimal control problem (5) with the correct value of θ_2.

Figure 4 depicts percentage welfare loss (with regard to the social optimum) as a function of actual heterogeneity. The simulation suggests that adverse welfare impacts increase with agent heterogeneity, potentially reaching high levels.

Conclusion

Previous literature on strategic behavior among users of an aquifer has abstracted away from the complicating factors of imperfect transmissivity in groundwater flows and user heterogeneity. This paper provides a step forward by presenting an analytical framework

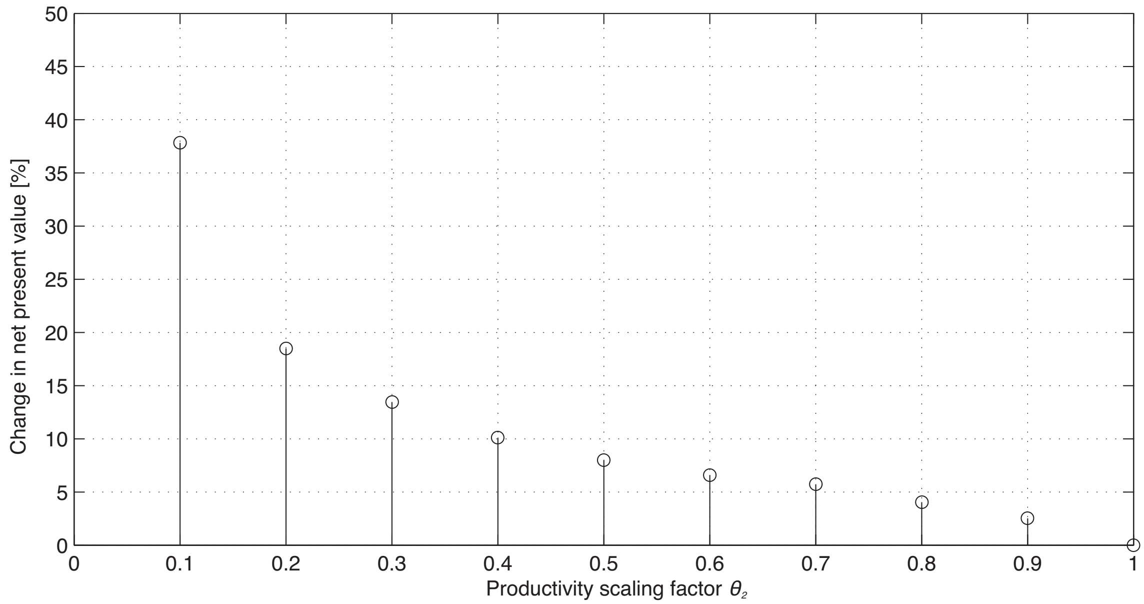

Figure 4: Percent social welfare loss from incorrect farm homogeneity assumption.

for dealing with both these issues in cases where pumping costs are external to the user. Although results generalize to an arbitrary number of aquifer cells and agents, we illustrate the importance of properly accounting for these factors with a simple two-cell, two-agent example.

Moreover, unlike previous work, we present a mechanism for inducing a socially optimal extraction path in Markov perfect equilibrium. In spite of the underlying complexity of the system, and the government's inability to monitor individual user extraction, the optimal mechanism is quite simple: a dynamic linear tax/surcharge based upon the height of a user's (potentially shared) local water table. This result holds irrespective of the aquifer's physical characteristics (number of cells, transmissivity) as well as the number and spatial configuration of extracting agents. An additional contribution of our model rests on the notable similarities in structure between our diffusive mutil-cell aquifer model and stock non-point source pollution. Thus, our results may provide insights for developing dynamic ambient-based policy instruments for a non-uniformly mixed pollutant.

Like any modeling exercise, the approach here relies on a set of simplifying assumptions that suggest both caveats and potentially fruitful courses of future research. For example, while the assumption of external pumping costs may be appropriate for many developing countries (notably India), it is clearly not in many industrialized countries. Although our policy does not rely on monitoring individual extraction rates of users sharing a micro-watershed, it does assume costless continuous monitoring of deterministic water table levels. An interesting avenue of research would be to determine the welfare implications arising from imperfect monitoring in both space and time (e.g., if the regulator can only check water table levels at a subset of locations at discrete intervals) and stochastic flows.

Appendix

Extension to Multiple Agents. Let there be N agents indexed by $n = 1, 2, ..., N$, and let $S_n \subseteq \{1, 2, .., N\} \equiv S$ denote agent n's neighbors. The degree to which neighboring agents n and m are connected is denoted by $c_{nm} = c_{mn}$ where $c_{nm} \in [0, \infty]$ (we set $c_{nn} = 0$). Generalizing the dynamics of Eq. (1), the water table of agent of agent n obeys the following differential equation[12]

$$a_n \dot{x}_n = r - q_n + \sum_{j \in S_n} c_{nm}[x_m - x_n]. \tag{30}$$

In a multiple-agent environment, the vectors $\mathbf{x}$ and $\mathbf{q}$ are extended in the obvious way and the social cost function is generalized to $D(\mathbf{q}, \mathbf{x})$. The socially optimal solution given by the optimization problem (5) is also appropriately modified.

In the context of linear mechanisms, it is possible to adapt Proposition 1 to give us insight into the equilibrium behavior of the system. In particular, given an agent n and a mechanism $\boldsymbol{\phi}$, an interior Markov-perfect equilibrium in open-loop strategies q_n^* satisfies

$$\frac{\mathrm{d}h_n(q_n^*(t))}{\mathrm{d}q_n} - \frac{\lambda_n^{n,\boldsymbol{\phi}}(t)}{a_n} \quad = \quad 0.$$

where $\lambda^{n,\boldsymbol{\phi}}(t)$ solves the following system of differential equations with terminal conditions:

$$\begin{aligned} &\dot{\boldsymbol{\lambda}}^{\boldsymbol{n,\phi}} = \mathbf{A} \cdot \boldsymbol{\lambda}^{n,\boldsymbol{\phi}}(t) + \mathbf{b}^n(t) \\ &\lambda_n^{n,\boldsymbol{\phi}}(T) = \beta_n^T, \;\; \lambda_m^{n,\boldsymbol{\phi}}(T) = 0 \;\text{ for all } m \neq n. \end{aligned} \tag{31}$$

Here $\mathbf{A} \in \Re^{N\times N}$ and $\mathbf{b}^n \in \Re^N$ are such that

$$\mathbf{A}_{kk} = \delta + \frac{1}{a_k}\sum_{j\in S_k} c_{kj} \quad \text{for all } k \in \{1,2,..,N\}$$
$$\mathbf{A}_{kj} = -\frac{c_{kj}}{a_m}, \quad \text{for all } \; k \in \{1,2,...,N\}, \;\; j \in S_k$$
$$\mathbf{A}_{kj} = 0, \quad \text{for all } \; k \in \{1,2,...,N\}, \;\; j \notin S_k$$
$$\mathbf{b}^n_n = -\beta_n(t), \;\; \text{and} \;\; \mathbf{b}^n_m = 0 \;\; \text{otherwise.}$$

Once a solution to system (31) is obtained, it can be used to influence Markov-perfect equilibrium behavior along the lines of Theorem 1. In particular, given an arbitrary feasible, continuously differentiable, N-dimensional extraction path, we can show that a mechanism ϕ given by Eq. (10) exists, which induces it in Markov-perfect equilibrium.

Theorem 2 *Theorem 1 extends to the N-agent case.*

Proof. Similarly to the proof of Theorem 1, we wish to find $\boldsymbol{\phi}$ so that

$$\frac{\mathrm{d}h_n(\hat{q}_n(t))}{\mathrm{d}q_n} = \frac{\lambda_n^{n,\boldsymbol{\phi}}(t)}{a_n}, \;\; \text{for all } t \in [0,T]. \tag{32}$$

A general solution for the system of linear differential equations given by Eqs. (31) can be found in Chapter 2.3.4 of Coddington and Carlson (1997):

$$\boldsymbol{\lambda}^{n,\boldsymbol{\phi}}(t) = \boldsymbol{\Lambda}^n(t)\boldsymbol{\xi} + \boldsymbol{\Lambda}^n(t)\int_0^t \left[\boldsymbol{\Lambda}^n(s)\right]^{-1}\mathbf{b}^n(s)\mathrm{d}s, \;\; t \in [0,T], \tag{33}$$

where $\boldsymbol{\xi} \in \Re^n$ and $\boldsymbol{\Lambda}^n(t)$ is a basis for the solutions to the homogeneous counterpart of system (31). Performing the change of variable $z = T - t$, choosing $\boldsymbol{\Lambda}^n$ so that $\boldsymbol{\Lambda}^n(z) = \mathbf{I}_n$ at $z = 0$, and setting $\boldsymbol{\xi}$ to a vector $\boldsymbol{\xi}^{\beta_n^T}$ such that the transversality conditions in Eqs. (31) are satisfied,[13] obtains the following unique solution of system (31)

$$\boldsymbol{\lambda}^{n,\boldsymbol{\phi}}(z) = \boldsymbol{\Lambda}^n(z)\boldsymbol{\xi}^{\beta_n^T} - \boldsymbol{\Lambda}^n(z)\int_0^z \left[\boldsymbol{\Lambda}^n(s)\right]^{-1}\mathbf{b}^n(T-s)\mathrm{d}s, \;\; z \in [0,T]. \tag{34}$$

Denote row m of matrix $\boldsymbol{\Lambda}^n$ by $\boldsymbol{\Lambda}^n_m$. The restriction of vector (34) to coordinate n obtains

$$\boldsymbol{\lambda}_n^{n,\boldsymbol{\phi}}(z) = \boldsymbol{\Lambda}^n_n(z)\boldsymbol{\xi}^{\beta_n^T} - \int_0^z \left[\boldsymbol{\Lambda}^n(z)\left[\boldsymbol{\Lambda}^n(s)\right]^{-1}\right]_{nn}\beta_n(T-s)\mathrm{d}s, \;\; t \in [0,T], \;\; z \in [0,T]. \tag{35}$$

Using Eq. (35), we adapt condition (25) to obtain the following Volterra integral equation of the first kind

$$-a_n\left[\frac{\mathrm{d}h_n(\hat{q}_n(T-z))}{\mathrm{d}q_n}\right]+\boldsymbol{\Lambda}^n_n(z)\boldsymbol{\xi}^{\beta^T_n}= \tag{36}$$
$$\int_0^z\left[\boldsymbol{\Lambda}^n(z)\big[\boldsymbol{\Lambda}^n(s)\big]^{-1}\right]_{nn}\beta_n(T-s)\mathrm{d}s,\quad \text{for all } z\in[0,T].$$

We set β^T_n so that Eq. (37) is satisfied for $z=0$. The integral equation's kernel

$$\Theta(z,s)=\left[\boldsymbol{\Lambda}^n(z)\big[\boldsymbol{\Lambda}^n(s)\big]^{-1}\right]_{nn}$$

is such that $\Theta(z,z)=1$. This fact, in combination with our differentiability assumptions, implies that Eq. (37) may be reduced to the following equivalent Volterra integral equation of the second kind

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(-a_n\left[\frac{\mathrm{d}h_n(\hat{q}_n(T-z))}{\mathrm{d}q_n}\right]+\boldsymbol{\Lambda}^n_n(z)\boldsymbol{\xi}^{\beta^T_n}\right)= \tag{37}$$
$$\beta_n(T-z)+\int_0^z\frac{\mathrm{d}}{\mathrm{d}z}\Theta(z,s)\beta_n(T-s)\mathrm{d}s,\quad z\in[0,T].$$

Our continuity and differentiability assumptions ensure that Theorem 2.1.1 in Burton (2005) applies and integral equation (38) has a unique solution. Repeating the argument for all agents establishes the result. ■

Notes

[1]See Brozović et al. (2006) for a thorough discussion of these issues.

[2]Assuming homogeneous agents greatly simplifies the analysis. All agents behave symmetrically, extracting the same amount of water in each period. There are thus no horizontal flows and hydrology is irrelevant.

[3]For expository reasons, we limit attention to one agent per cell, so the two terms are

interchangeable. In such cases, it is theoretically possible to infer the pumping schedules of agents from the evolution of the water tables. Under the more general setting derived in the appendix, however, such inference is not possible.

[4]More generally, with multiple users per cell, this equation would read $a_i \cdot \dot{x}_i = r - \sum_{k \in A_i} q_k + c[x_j - x_i], \;\; i = 1,2 \;\; i \neq j$. Here, A_i denotes the set of agents occupying cell i.

[5]Our analysis extends to cases where recharge rates are different and may depend linearly on water tables.

[6]Using water balance and connectivity between individual cells with uniform recharge rates to model flows is a simple version of the finite difference discretization methods commonly used in the hydrological literature to simulate existing aquifers with complex geometry and boundary conditions Harbaugh (2005). The $n \geq 2$ player version of our approach, detailed in the appendix, readily extends to these more general models.

[7]In our discussion of the differential game, we do not explicitly impose the state nonnegativity constraint, $\mathbf{x}(t) \geq \mathbf{0}$. A similar approach is common in the groundwater economics literature. Water-table nonnegativity is typically not explicitly imposed (Gisser and Sánchez, 1980; Fisher and Rubio, 1997; Zeitouni and Dinar, 1997; Roseta-Palma and Xepapadeas, 2004; Aggarwal and Narayan, 2004; Brozović et al., 2006, among others), or modeled asymptotically allowing for finite (or even steady-state) violation at some parameter values (e.g., Rubio and Casino, 2003). Other studies either impose structure on model primitives that precludes socially optimal corner solutions (Negri, 1989), or deal with static water-table levels (Chakravorty and Umetsu, 2003).

[8]See Chapter 4 of Dockner et al. (2000) for a detailed discussion of the role of informational assumptions in the determination of equilibrium strategies.

[9]See Raj (2004) for data on climate and groundwater, Shiferaw et al. (2008) for crop and agricultural production specific data, and World Bank (2001) plus references therein for energy data.

[10]This specification implicitly assumes rainfed agricultural production is infeasible.

[11]In this context, agents set their pumping rates to q_i such that $\mathrm{d}h_i(q_i)/\mathrm{d}q_i = 0$.

[12]When $c_{nm} = \infty$ agents n and m share the same cell and their interaction is described by the multiple-agent equivalent of the dynamics that appear in Footnote 4. To avoid cumbersome notation we remain consistent with our previous analysis and assume that all n agents are in different cells.

[13]Since the matrix $\mathbf{\Lambda}^n$ has full rank, $\boldsymbol{\xi}^{\beta_n^T}$ exists and is uniquely determined.

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